FINITE SIMPLE GROUPS AND THEIR CLASSIFICATION[†]

BY

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Introduction

The subject of finite simple groups appears to be very forbidding to the mathematical community at large. Papers run to impossible lengths, arguments look ad hoc, methods are seemingly unrelated to those in the rest of algebra. As if this were not bad enough, the whole field abounds with pathologies—strange new monsters called *sporadic* groups, twenty of them at last count—and even more absurd, some of these creatures depend on the calculations of a high speed computer for a demonstration of their very existence. At moment of writing, such a computer (hopefully) is giving birth to sporadic group #21[‡]. Moreover, the theorems give no clue where the next group is to be found, nor any indication that the situation is about to improve. Obviously under such conditions the wisest thing for any mathematician to do is to keep as far away from the finite simple groups as he possibly can.

Yet simple groups constitute a topic of intrinsic mathematical interest. The concept of a group is certainly one of the most natural and fundamental in all of mathematics, and the finite groups themselves have a venerable history going back to Galois' proof of the insolvability of the quintic equation by radicals, which depends ultimately on the group-theoretic nonsolvability of the symmetric group of degree 5. And the simple groups are the fundamental building blocks for all finite groups. Hence the problem of determining all simple groups is a natural one of broad mathematical appeal.

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[‡] The number of sporadic groups has now reached 23.

Of course, even if we knew all the simple groups, we would be far from knowing everything about finite groups. After all, we certainly understand the groups of prime order, yet we have a great deal to learn about solvable groups: the family of groups which one can construct out of the groups of prime order. Likewise, basic questions about the known simple groups remain unanswered. For example, we are unable, in general, to write down their character tables or even to give an algorithm for doing so. Nevertheless, for the foreseeable future the central question concerning finite groups which is of widest mathematical interest is the problem of finding all the simple groups.

I would like to present a roadmap of the situation at the present time, what results have been obtained with what methods, and at the end give some idea of the prospects for the future. There will be no attempt to prove anything at all, just a leisurely journey over the whole landscape to give the reader some feeling about the entire field. Hopefully when I have finished you will see that the subject is not really chaotic, that some orderly progress has taken place and some general methods for attacking problems have been developed. At the same time I think I can explain to you how it happens that the papers turn out to be as long as they do.

To some extent this paper will constitute an up-to-date treatment of what I attempted, with somewhat more detail, in [2, Chapters 16 and 17] which described the state of affairs in 1968. Walter Feit's survey article [1] has the same general objective. In addition, my 1969 lectures at Oxford, published in [2, Chapter II], gives a largely expository treatment of the whole area of centralizers of involutions in simple groups. These three references will provide added knowledge and understanding of the various topics to be discussed here.

To begin with, since we are so far from determining all simple groups, we ask the following kind of questions:

What are all the finite simple groups having some property X?

Obviously the more restrictive we make X, the greater chance we have of answering the question. On the other hand, as we broaden X, the closer we approach the ultimate question: What are all the finite simple groups?

When one asks a question in mathematics, the first thing one needs is a conjecture, something that one thinks might be true. In many fields finding the right conjecture is the major task; once one has formulated the proper answer to the question, one can often find a proof of it without too much difficulty. In finite group theory, the situation is completely reversed. The proper conjecture is something that one can arrive at immediately. Namely, go through the list of all the known simple groups and see which ones have property X and then conjecture that these are the only simple groups having that property.

Of course, conjectures are often wrong—and perhaps there is an as yet unknown simple group having property X. This doesn't matter—we have still made the right conjecture to begin with. Because the nature of our proof is the following: We always consider a counterexample of least order. Thus, we always take a simple group G having property X such that any simple group of lower order with property X is among the known simple groups. Our aim, of course, is to prove that no such minimal counterexample exists, and we begin to argue on our group G to that end. But suppose a counterexample does exist—in other words, suppose our conjecture is false. Will our effort be wasted? The answer is no, for everything we establish about G will amount to a property of an unknown simple group of least order having property X—and it may very well happen that we shall determine enough properties of G to enable us eventually to establish the existence of such a new simple group. Historically this method of discovering new simple groups has occurred several times.

Let me make one other observation. Frequently, in fact usually, it happens that G "involves" many simple groups with property X. What I mean by this is that there exist subgroups H, K of G with K normal in H and H proper in G such that H/K is a simple group with property X. Then H/K has lower order than G and so is a known simple group. This will be an important fact to us, for we can then use properties of the known group H/K to deduce properties of H and eventually, properties of G. The upshot of this discussion is that one certainly ought to have a good knowledge of the known simple groups including their basic properties before undertaking any investigation of finite simple groups.

Thus in the first part of this survey we shall give a brief description of the known simple groups. Only then will we be ready to consider classification problems concerning simple groups. We shall divide these into two types: *special* and *general* classification problems. In Part II we shall discuss special classification problems, subdividing them into what we call three *Levels*: Level I consists of *recognition* theorems, Level II of characterizations in terms of the structure of the *centralizers of involutions*, and Level III of characterizations in terms of the structure of a *Sylow 2-subgroup*. The use of the word *level* here is deliberate, for as we shall see the solution of a problem in a given level requires the solution of one or more problems from prior levels. Moreover, from this point of view, general classification problems, which we discuss in part III, should be considered as Level IV. Finally in Part IV of this survey, we shall give some personal views about the prospects for classifying all finite simple groups.

I. The Known Simple Groups

1. Alternating and classical groups. At the outset, one has the cyclic groups Z_p of prime order. From the point of view of simple group classification, these are regarded as trivial simple groups and are not even considered. Thus simple group really means *nonabelian* simple group.

The finite simple groups with which the reader is most likely familiar are the alternating groups A_n of degree $n \geqq 5$. These are the subgroups of index 2 in the symmetric groups S_n of degree n consisting of all even permutations.

Next come the analogues of the classical groups: the linear, symplectic, orthogonal, and unitary groups. These are essentially groups of matrices, but now with elements from the finite field $GF(q)$ with q elements. Thus the *general linear group* $GL(n, q)$ consists of all $n \times n$ nonsingular matrices with entries from $GF(q)$. $SL(n,q)$, the *special linear group*, is the subgroup of $GL(n,q)$ consisting of those matrices of determinant one. Then one defines $PGL(n,q)$ as the factor group of $GL(n,q)$ modulo its center, which consists of the group of nonsingular scalar matrices. This is the *projective general linear group* and the image $PSL(n,q)$ of $SL(n,q)$ is the *projective special linear group.* It turns out that if $n > 2$ or $q > 3$, $PSL(n,q)$ is a simple group. In the remaining cases the groups are actually solvable, which is analogous to what happens in the case of alternating groups of degree 2, 3, or 4.

Similarly one defines *symplectic* and *projective symplectic groups* over $GF(q)$ as those preserving some nondegenerate alternating bilinear form defined on an n-dimensional vector space over $GF(q)$. One must be a little careful if q is a power of 2 as then an alternating form is symmetric. Likewise *orthogonal groups* are defined relative to nondegenerate symmetric bilinear forms and just as in the classical case there are two types of forms. I shall not attempt to say more than this.

For our purposes, the best way to define a unitary matrix is as follows. If we consider $GL(n,C)$, where C represents the complex numbers, then the mapping $X^\alpha = ((\bar{X})^t)^{-1}$ for X in $GL(n,C)$ determines an automorphism of $GL(n,C)$. Here $\bar{X}$ denotes the complex conjugate of X and t denotes the transpose. Then the matrix X is *unitary* if and only if $X^\alpha = X$. Thus the classical unitary matrices are just the matrices fixed by α. The analogue of this is to consider the group $GL(n, q^2)$. Then there exists a unique automorphism σ of $GF(q^2)$ over $GF(q)$ of period 2 which sends each element of $GF(q)$ to its qth power and, for X in $GL(n, q^2)$, $\bar{X}$ means simply the matrix obtained by replacing each entry of X by its image under σ. Then again if α is defined by $X^\alpha = ((\bar{X})^t)^{-1}$, a matrix is called *unitary* if $X^\alpha = X$. This defines the *general unitary group* $GU(n, q)$ (sometimes denoted by $GU(n, q^2)$) and again we obtain simple groups by considering *projective unitary groups.*

2. Groups of Lie type. The classical groups are particular cases of Lie groups. In the 19th century, Elie Cartan completely classified all connected complex Lie groups by classifying all complex simple Lie algebras, the Lie groups arising as groups of automorphisms of their associated algebras. In his notation, there are four infinite familes of simple algebras, denoted by A_n, B_n, C_n, and D_n and, apart from these, precisely five exceptional algebras, denoted by G_2, F_4, E_6, E_7, E_8. Here the subscript designates the dimension of a Cartan subalgebra. (Do not confuse A_n here with the alternating group A_n.) Using the same notation for the corresponding groups, A_n corresponds to $SL(n+1, C)$. Likewise C_n corresponds to a suitable symplectic group, and B_n, D_n to suitable orthogonal groups. Thus we have finite analogues of the groups A_n, B_n, C_n, D_n, which in the Lie notation we designate as $A_n(q)$, $B_n(q)$, etc.

In the early years of the twentieth century Dickson determined finite analogues of the groups G_2 and E_6, but it was not until 1955 that Chevalley, in a celebrated paper, gave a unified treatment of the entire subject and showed that analogues of every one of these groups existed over every field K and, in particular, over $GF(q)$. Thus we have groups $G_2(q)$, $F_4(q)$, etc.

Just as the unitary groups were defined relative to the family A_n, so it was known that the families D_n and E_6 admitted similar automorphisms and, if one considered the fixed points relative to these automorphisms, one obtained new simple groups. Moreover, Steinberg has shown that Chevalley's approach carries through for these groups and leads to finite analogues of the corresponding Lie groups.

The Chevalley-Steinberg groups, together with the alternating groups, account for almost all of the known simple groups. But actually the story of the analogues of groups of Lie type does not end here. I must now mention some work of Suzuki which will be our first example of a classification problem. An involution is simply the name of an element of order 2 in a group. Around 1960 Suzuki was trying to determine all finite simple groups in which the centralizer of every involution was a 2-group—that is, had order a power of 2. Making the natural conjecture, he hoped to prove that, except in certain low dimensional cases, the only groups were $PSL(2,2^n)$ and $PSU(3,2^n)$, $n \geqq 2$. However, there was one configuration that he was led to that would not succumb. He could not reach the desired contradiction. In the end, he began to suspect that there must be some real groups that satisfied his conditions. Finally in a stroke of sheer genius he was able to write down certain 4×4 matrices over $GF(2^n)$, n odd, which generated a group that satisfied his conditions and thus he discovered the family $Sz(2^n)$ of simple groups that bears his name.

Let me state Suzuki's definition of his groups, as this will give some feeling of their description. Let $q = 2^{2m+1}$ with $m \geqq 1$ and let θ be the automorphism of $GF(q)$ such that $\theta^2 = 2$—that is, θ^2 is the squaring map in $GF(q)$. Because q is an odd power of 2, there exists a unique such automorphism θ.

Let S and H denote, respectively, the set of matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^{1+\theta}+b & a^{\theta} & 1 & 0 \\ a^{2+\theta}+ab+b^{\theta} & b & a & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c^{1+\theta^{-1}} & 0 & 0 & 0 \\ 0 & c^{\theta^{-1}} & 0 & 0 \\ 0 & 0 & c^{-\theta^{-1}} & 0 \\ 0 & 0 & 0 & c^{-1-\theta^{-1}} \end{bmatrix}$$

where $a, b, c \in GF(q)$ and $c \neq 0$; and let w be the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then S is a 2-group of order q^2, H is a cyclic group of order $q-1$ which normalizes S, and w is an involution. By definition,

$$Sz(q) = \langle S, H, w \rangle.$$

Suzuki showed that for each q, $Sz(q)$ is a simple group of order $q^2(q-1)(q^2+1)$.

At this point Suzuki was able to achieve his objective of classifying all simple groups with the given centralizer property, but now his answer read $PSL(2,2^n)$,

$PSU(3, 2^n)$, or $Sz(2^n)$. Suzuki's discovery produced an extra dividend as his groups provided the first and only examples of simple groups of order relatively prime to 3.

Now Rhimak Ree comes on stage. He took one look at Suzuki's 4×4 matrices and saw that they were closely related to the family $B_2(2^n)$. In general, the family $B_2(q)$ does not possess the kind of automorphism that the families $A_n(q)$, $D_n(q)$, and $E_6(q)$ do that allow one to make the Steinberg construction. However, it was known that just in characteristic 2, that is, when $q = 2^n$, $B_2(q)$ possessed an *extra* automorphism. Moreover, Ree observed that if one plays the Steinberg game with respect to this automorphism, one ends up with precisely the Suzuki groups. But Ree also knew that there were two other situations in which extra automorphisms existed: namely, $G_2(3^n)$ and $F_4(2^n)$, n odd. So he immediately made the corresponding constructions in these cases and was led to two further families of simple groups, known respectively as the Ree groups of characteristic 3 and 2.

The Suzuki and Ree groups together with those of Chevalley and Steinberg constitute the finite simple groups of *Lie type.*

3. Sporadic groups. Apart from the alternating groups and the groups of Lie type, there exist at present only *twenty* other known simple groups. These are the so-called *sporadic* groups which do not appear to be a part of any infinite family of simple groups.

I wish to describe these to you now. As we shall see, their discoveries have not been as haphazard as might appear, but, as in the case of the Suzuki groups, many of them are intimately connected with classification problems.

However, the first five of these groups were discovered a long time ago, in the early 1860's, to be exact, by Mathieu. These groups, denoted by M_{11}, M_{12}, M_{22}, M_{23} and M_{24} are certain multiply-transitive permutation groups on the corresponding numbers of letters. To this day, M_{12} and M_{24} are the only known simple groups, aside from the alternating groups themselves, that act quintuply transitively on some finite set. Mathieu really discovered only two groups, namely, M_{12} and M_{24}; the remaining groups arise as subgroups of these. Thus M_{11} is just the stabilizer of a point in M_{12}, M_{23} the stabilizer of a point in M_{24}, and M_{22} the stabilizer of a point in M_{23}. A similar phenomenon occurs in two other situations where certain subgroups of a newly constructed simple group themselves turn out to be new simple groups.

It is remarkable that no further sporadic groups were discovered during the 100 years following Mathieu. The next point in our story begins precisely where we left off, namely with the Ree groups of characteristic 3, which we denote by $R(3^n)$. The groups $R(3^n)$ are very interesting because their Sylow 2-subgroups are abelian—in fact, elementary abelian of order 8. Moreover, all involutions of $R(3^n)$ are conjugate and every involution of the group has a centralizer isomorphic to $Z_2 \times PSL(2, 3^n)$. Because of this, it was very natural to try to determine all simple groups with elementary abelian Sylow 2-subgroups of order 8 in which the centralizer of an involution is isomorphic to $Z_2 \times PSL(2, p^n)$ for some odd prime p. Again the goal was to prove that such a group had to be isomorphic to $R(3^n)$. Clearly as a first step, one would want to prove that the prime p had to be 3. Now, the analysis that leads to this conclusion forces p to satisfy a certain diophantine equation. It turned out that, apart from the general solution $p = 3$, n an arbitrary odd integer, there was one further solution: namely, $p^n = 5$. This is very often the case in finite group theory—certain arithmetic situations lead to general solutions plus one or more exceptions involving low values of the parameters. It is partly out of this experience that many group theorists feel that the sporadic groups are essentially small phenomena. Of course, *small* here is a very relative term. Since the 19th sporadic group happens to be a permutation group on approximately 8,800,000 letters, one might not tend to regard it as so small!

Returning to the case $p^n = 5$, it was Janko who pushed this through to completion and discovered a new simple group J_1 of order 175,560. Eventually he proved enough things about a possible group having the given properties to suggest that he write down two explicit 7×7 matrices with entries in $GF(11)$: namely,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 & 2 & -1 & -1 & -3 & -1 & -3 \\ -2 & 1 & 1 & 3 & 1 & 3 & 3 \\ -1 & -1 & -3 & -1 & -3 & -3 & 2 \\ -1 & -3 & -1 & -3 & -3 & 2 & -1 \\ -3 & -1 & -3 & -3 & 2 & -1 & -1 \\ 1 & 3 & 3 & -2 & 1 & 1 & 3 \\ 3 & 3 & -2 & 1 & 1 & 3 & 1 \end{bmatrix}.$$

He was then able to demonstrate that these two matrices generated a simple group of the given order with the given properties. In addition, he was able to prove that this was the only such group.

Before going on, I should remark that it is still an unsettled question whether the groups $R(3^n)$ are the only simple groups in the case that $p = 3$ (except for the values $n = 3, 5$, or 7). Thompson has spent close to two years on this problem, has made considerable progress, but as yet has been unable to complete the analysis. I shall return to this later when I put this open question in the broader context of the whole area of what we call recognition theorems.

Here, then, was at least a procedure for trying to discover new simple groups: study groups in which the centralizer of an involution was either identical with or closely resembled that in some known simple group. Preferably these centralizers should be fairly small—in keeping with the principle that pathological phenomena occur only in the "small". Again it was Janko who struck gold! Beginning with a single centralizer—a certain extension of a group of order 32 by A_5—he showed that there might actually be *two* groups with such a centralizer. In the end, he needed some assistance in proving their existence. Marshall Hall and Wales proved the existence of the one of lower order, and Graham Higman and McKay the one of larger order. So now we have groups J_1, J_2, and J_3. The largest Janko group J_3 has order about 50,000,000.

The Janko group J_2 turned out to be very important, for its construction led ultimately to *four* further sporadic groups, the last of these being the most recently discovered such group, the Rudvalis group. It will be preferable to describe the properties of J_2 which led to these four groups before continuing our centralizer of involution story. J_2 has a maximal subgroup $H \cong PSU(3,3)$ of index 100; and if one considers the permutation representation of J_2 on the right cosets of H, one obtains a primitive permutation representation of J_2 of degree 100 in which H is the stabilizer of a point and the action of H on the remaining 99 points breaks up into orbits of lengths 36 and 63, respectively. Thus H has *three* orbits on the initial 100 cosets and so J_2 is an example of a primitive *rank* 3 permutation group. Note that if H had had a single orbit on the remaining letters, J_2 would have been a doubly transitive permutation group. Thus rank 3 permutation groups represent the first degree of generalization of doubly transitive permutation groups.

Wielandt and Donald Higman had developed a general theory of rank 3 permutation groups which Hall and Wales well understood, and it was by means of this theory that they proved the existence of J_2. One can describe the construction best perhaps in terms of planar graphs. The problem comes down to constructing such a graph with 100 vertices with the following properties:

a) Designating one of the vertices by ∞, if one deletes ∞ plus all lines emanating from it, the remaining graph should have $PSU(3,3)$ as its automorphism group; and

b) The entire graph should possess an automorphism in which the vertex ∞ is not left fixed.

Shortly after the construction of J_2, Hall gave a lecture on it at Oxford. Donald Higman and Sims were in the audience. They were both struck by the fact that the group M_{22} had certain permutation properties analogous to those of $PSU(3,3)$, and that very same night (!!) they proceeded to construct a new primitive rank 3 permutation group in which M_{22} was a maximal subgroup, again of index 100.

Since then three further simple primitive rank 3 permutation groups have been constructed: one by McLaughlin of degree 275 in which the stabilizer of a point is $PSU(4,3)$; one by Suzuki of degree 1782 in which the stabilizer of a point is $G_2(4)$; and the most recent Rudvalis group of degree 4060 in which the stabilizer of a point is the Ree group constructed from the group F_4 over the prime field with 2 elements.

The reason I wished to describe the five primitive rank 3 sporadic groups is that one of these, McLaughlin's group, in turn suggested a centralizer of an involution problem which ultimately gave rise to the nineteenth sporadic group. In his group all involutions are conjugate and the centralizer of any one of them is isomorphic to a nonsplit perfect central extension of Z_2 by A_8. This group is designated $\hat{A}_8$. Analogous extensions $\hat{A}_n$ exist for all the alternating groups (for $n \geq 5$) and so it was natural to consider whether there are other simple groups in which the centralizer of some involution was isomorphic to $\hat{A}_n$ for suitable n. For $5 \leqq n \leqq 7$, one knew from a theorem of Brauer and Suzuki that no such simple group could exist inasmuch as such a group would necessarily have generalized quaternion Sylow 2–subgroups. When $n = 8$, one could show that McLaughlin's group was the only possible one. Janko considered the cases $n = 9$ and 10 and showed that there were no simple groups with such a centralizer of an involution. At that point he lost interest.

However, this was a mistake, as hindsight reveals, for the groups $\hat{A}_{10}$ and $\hat{A}_{11}$ have very interesting Sylow 2–subgroups, which are isomorphic to each other. These groups, of order 2^8, had already appeared in the Ph.D. thesis of Anne MacWilliams, a student of Thompson's, as an exceptional case. In fact, she had been unable to rule out such a 2–group as a possible candidate for a Sylow 2–subgroup of a simple group. As a consequence, Thompson suggested to Richard Lyons that it might be worthwhile to examine the case of $\hat{A}_{11}$. This Lyons did (along with the $\hat{A}_{10}$ case which Janko had not written up in detail) and eventually his investigations together with subsequent work of Sims, resting ultimately on computer calculations, proved the existence of a unique simple group with such a centralizer of an involution.

Existence here turned out to be a major task since the permutation representation being sought had to be on approximately 8,800,000 letters and the resulting group had to be a primitive rank 5 permutation group with $G_2(5)$ as the stabilizer of a point. In addition to this, Thompson himself showed that when $n \geqq 12$, no simple groups exist in which the centralizer of an involution is $\hat{A}_n$, thus putting fini to this line of investigation.

One other centralizer of an involution problem has led to a new simple group. It was known that the groups $PSL(5,2)$ and M_{24} possessed involutions with isomorphic centralizers. Thus it was of interest to try to prove that these were the only simple groups having such a centralizer. It was Held who investigated this situation and surprisingly it has turned out that yet another group besides $PSL(5,2)$ and M_{24} has the same centralizer of an involution! This is Held's simple group.

We have now accounted for all but six of the sporadic simple groups, the three Conway groups and the three Fischer groups. In each case, all three groups arose out of a single problem. In Conway's case, there was a certain lattice known as the Leech lattice determined by a set of vectors in 24–dimensional Euclidean space with integral coordinates which, in some technical sense, was extremal among such 24–dimensional integral lattices. Conway studied its automorphism group. Although this group is not simple, within it lay three new simple groups, the Conway groups. Remarkably it contains as well several of the previously constructed sporadic groups.

On the other hand, Fischer was investigating the following interesting group-theoretic problem, suggested by the symmetric group S_n. S_n is clearly generated by all its transpositions — that is, 2–cycles (ij), $i \neq j$, $1 \leqq i, j \leqq n$. Moreover, the product of any two transpositions is immediately seen to be either the identity, a product of two disjoint 2–cycles, or a 3–cycle and so such a product necessarily has order 1, 2, or 3. Fischer considered the general problem of determining all finite groups which can be generated by a conjugacy class of involutions in which the product of any two members of the class has order 1, 2, or 3. His complete solution of this problem led to the existence of the three sporadic groups that bear his name.[†]

[†] Recently, Fischer has considered various generalizations of this problem. Out of this work has come sporadic groups #22 and # 23 and the very strong likelihood of yet two further sporadic groups.

This completes the run-down of all presently known sporadic groups apart from O'Nan's possible group†, the existence of which Sims is trying to demonstrate at this time, again with the aid of a computer. Let me say a few things about the very interesting situation which has given rise to this group. Several years ago, Alperin and I were investigating finite groups of 2–rank 3, (For any prime p, the *p-rank* of a group is, by definition, the maximal rank in the ordinary sense of an abelian p-subgroup of the group. Thus a group of 2–rank 3 contains a subgroup isomorphic to $Z_2 \times Z_2 \times Z_2$, but none isomorphic to $Z_2 \times Z_2 \times Z_2 \times Z_2$). In this connection, Alperin studied the nontrivial extensions of $Z_{2^n} \times Z_{2^n} \times Z_{2^n}$ by $GL(3,2)$, the latter group being, of course, the automorphism group of $Z_2 \times Z_2 \times Z_2$; and he was able to show that for each positive value of n, there exists a unique such *split* extension and a unique such *nonsplit* extension.

The pertinent question is, then, which such extensions can arise as 2–local subgroups of a simple group? (By definition, the normalizer of a nonidentity p-subgroup of a group, p a prime, is called a *local* subgroup of that group. If one wishes to specify the prime, one says *p-local* subgroup). Now when $n = 1$, it is well known that both the split and nonsplit extensions occur as 2–local subgroups of some simple group. Furthermore, it turns out that the Higman-Sims group contains a 2–local subgroup which is a split extension of $Z_4 \times Z_4 \times Z_4$ by $GL(3,2)$, this extension containing, in fact, a Sylow 2–subgroup of the Higman-Sims group.

Both Alperin and I had tried to interest people in the general problem; but it was not until O'Nan actually needed a particular case of the general problem for his work on doubly transitive permutation groups that anyone seriously attempted to investigate the general question. O'Nan has been able to show that when $n \geqq 3$, no simple group exists having such a split or nonsplit extension as a 2–local subgroup. However, in the nonsplit $n = 2$ case (that is, $Z_4 \times Z_4 \times Z_4$ by $GL(3,2)$, nonsplit), no contradiction ensued. Instead he was able to show that such a simple group, if it existed, necessarily had only one conjugacy cass of involutions and the centralizer of an involution was a uniquely determined group having a normal subgroup of index 2 which was a perfect central extension of Z_4 by $PSL(3,4)$. Thus he was, in fact, reduced to a centralizer of an involution situation. Ultimately he constructed the group order and the proper subgroup structure and has shown that the group, if it exists, will be a primitive rank 5 permutation group of degree approximately 600,000 in which the stabilizer of a point is isomorphic to the

† Sims has proved the existence of O'Nan's group; it is sporadic group # 21.

extension of $PSL(3, 7)$ by the automorphism of order 2 induced from the transpose-inverse map of $SL(3, 7)$.

A footnote: Held's group possesses a noncentral involution (that is, not in the center of a Sylow 2–subgroup) whose centralizer has a subgroup of index 2 which is a perfect central extension of $Z_2 \times Z_2$ by $PSL(3, 4)$, and so has a structure very close to that of O'Nan's centralizer. Held had previously considered whether there exist simple groups in which the centralizer of an involution had one of several structures closely resembling that in his own group; and in each case, after a considerable amount of effort, he showed that no such simple group exists. However, having no special motivation, he never considered O'Nan's centralizer. This shows the inherent problem of choosing the "right" centralizer of an involution to investigate!

Let me conclude this discussion with a few further remarks about the sporadic groups. As I have described their origins above, one can see that they have so far arisen in five essentially distinct ways:

a) As multiply transitive permutation groups;

b) From problems involving centralizers of involutions (and possibly, more generally, involving 2–local subgroups);

c) As primitive rank 3 permutation groups;

d) As automorphism groups of integral lattices;

e) From the analysis of certain types of conjugacy classes of involutions.

We have pointed out in connection with the $\hat{A}_n$ problem and again, just above, in connection with centralizer of involution problems, that some investigations have led to *non-existence* theorems. In actual fact, this has been the case more often than not. Obviously there are many interesting centralizers of involutions, many highly symmetric planar graphs, and some beautiful 48–dimensional and 72–dimensional integral lattices just waiting to yield a new simple group. I assure you, they have been sought out and investigated, but apart from the very few cases described above, have yielded only bitter fruit. Perhaps tomorrow someone will strike a rich harvest in one of these directions, but unfortunately there is something haphazard, even random, about this approach. Moreover, the next sporadic group may very well arise in some manner totally different from any of these.

As I shall indicate later, future research on general classification problems may provide a more systematic procedure in terms of internal group-theoretic properties for seeking out possible new simple groups.

II. Special Classification Problems

Classification problems fall into two major categories: *general* problems in which the assumptions carry over to all subgroups and homomorphic images of any group which satisfies the given conditions; and *special* problems, in which this is not the case. As we shall see, the solution of any general classification problem requires the prior solution of at least one and perhaps several special problems; therefore it will be best to discuss special classification problems first.

We should add that the borderline between the special and general is not so precise, for sometimes conditions which do not appear inductive become so with a slight reformulation. Hence there will be a certain degree of arbitrariness in our use of these terms. Another point is that special problems can themselves be considered to be of different degrees of generality, each stage requiring solutions from a preceding stage. To stress this dependency, we shall subdivide the special problems into *Levels* I, II, and III and shall then refer to general problems as *Level* IV.

1. Level I: recognition theorems. I have deliberately spent considerable time describing the known simple groups. Hopefully the survey will provide a sound base for the discussion of their classification that is to follow and which is our main concern in these talks.

As I remarked in the Introduction, a classification theorem concerning finite simple groups is just this: The determination of all simple groups G having some property X. The proof of any such theorem can always be organized in such a way that the last line of the argument reads:

Therefore G is isomorphic to one of the following groups... .

Here the dots are to be filled in by some set of known simple groups. This means that we must have some way of *recognizing* each of the known simple groups, or what amounts to the same thing, for each known simple group K we must be able to list some set of conditions which *characterize* K in the sense that if an abstract simple group G satisfies these conditions, then necessarily G is isomorphic to K.

The most common such characterization is in terms of a set of *generators and relations*. To illustrate, consider S_n and the transpositions $t_i = (i, i+1)$, $1 \leqq i \leqq n-1$. It is known that these $n-1$ transpositions suffice to generate S_n and, moreover, that the set of relations

$$(t_it_j)^{m_{ij}} = 1, \text{ where } m_{ij} = \begin{cases} 1 & \text{if } j = i; \\ 2 & \text{if } |i-j| \geqq 2; \\ 3 & \text{if } |i-j| = 1; \end{cases}$$

form a complete set of defining relations in the sense that S_n is isomorphic to the factor group of the free group generated by $x_1, x_2, \cdots, x_{n-1}$ modulo the normal closure of the subgroup generated by the corresponding elements $(x_i x_j)^{m_{ij}}$, $1 \leqq i, j \leqq n-1$. Taking t_i to be the image of x_i in this factor group, $1 \leqq i \leqq n-1$, we say that the t_i together with the given relations constitute a *presentation* of S_n.

Suppose now that one had an abstract group G which possessed a presentation given by generators t_i', $1 \leqq i \leqq n-1$, satisfying the same set of relations as the t_i's. Then it should be clear that the map $\alpha : t_i' \to t_i$, $1 \leqq i \leqq n-1$, extends in the natural way to give an isomorphism of G on S_n. Thus S_n is uniquely characterized up to isomorphism by the given presentation. In other words, S_n can be recognized by means of this presentation.

Each of the groups of Lie type as well as the alternating groups have natural presentations in terms of which they can be similarly recognized. Let me briefly explain the situation for the groups of Lie type, for the description will give you some feeling for the internal subgroup structure of these groups. It was Tits who first realized that the so-called Bruhat decomposition of algebraic groups could provide an axiomatic basis for describing and characterizing the finite groups of Lie type. It is embodied in the notion of a *(B, N)-pair*.

First, a group W is called a *Coxeter group* or a group *generated by reflections* provided it can be generated by involutions w_i, $1 \leqq i \leqq r$, such that for suitable integers k_{ij} the relations $(w_i w_j)^{k_{ij}} = 1$ form a complete set of defining relations for W; and, moreover, no proper subset of the w_i generate W. Thus S_n is a particular case of a Coxeter group. The integer r is called the *rank* of W. Coxeter has completely classified all such finite reflection groups, which accounts for the name Coxeter group.

Now let $G(q)$ be one of the groups of Lie type where $q = p^n$, p a prime. Let P be a Sylow p-subgroup of $G(q)$ and set $B = N_{G(q)}(P)$. Then P has complement H in B—that is, $B = PH$ with $P \cap H = 1$. Furthermore, there exists a subgroup N in $G(q)$ with the following properties:

a) $H = B \cap N$ is normal in N and $W = N/H$ is a Coxeter group;

b) G is a union of double cosets BuB with u in N;

c) If w_i, $1 \leqq i \leqq r$, are the defining involutions of W and u_i is a representative of w_i in N, $1 \leqq i \leqq r$, then for each u in N and each i, $1 \leqq i \leqq r$, we have

$$BuBu_iB \subseteq (BuB) \cup (Buu_iB);$$

d) $u_i^{-1}Bu_i \neq B$, $1 \leqq i \leqq r$.

For example, if we take $G(q)$ to be $GL(n, q)$, we can represent P as the set of all lower triangular matrices with all ones on the diagonal. Then H can be taken to be the set of diagonal matrices, N the set of all monomial matrices, and W can be identified with the set of all permutation matrices. Thus $W \cong S_n$, which is a Coxeter group of rank $n-1$. Properties (a)–(d) tell us something about the way that $GL(n, q)$ is generated by matrices of the given type, which, in effect, is a particular case of the way in which all groups of Lie type can be generated.

In general, the groups W, B, and H are called the *Weyl* group, a *Borel* subgroup, and a *Cartan* subgroup, respectively, of $G(q)$.

Actually Tits defines a (B, N)-pair by conditions that are weaker than those listed in (a)–(d). For example, it turns out that (b) is essentially a consequence of (c). Furthermore, in the general definition it is not assumed that B splits over a Sylow p-subgroup of G.

The group W dominates the structure of a (B, N)-pair G. Indeed, Tits has shown by what amounts to a very general geometric argument that if the rank of W is at least 3, then the structure of G is completely determined by W, and G is necessarily one of the groups of Lie type. More recently, Fong and Seitz, using group-theoretic arguments as well as many prior classification theorems have obtained the analogous result in the case that W has rank 2 and B splits in the manner described above.

A (B, N)-pair G in which W has rank 1 is simply a doubly transitive group in which B is the stabilizer of a point, for in this case $G = B \cup Bu_1B$, which implies that B acts transitively on the right cosets of B in G other than B itself. If we assume that $B = HP$ splits, as above, with H the stabilizer of two points, then P will act transitively and regularly on the cosets of B in $G - B$. A considerable amount of research has been devoted to the problem of determining all doubly transitive permutation groups of this general type: Zassenhaus, Feit, Suzuki, Ito, Glauberman, O'Nan, H. N. Ward, and Thompson have all worked on some aspect of the problem. A fairly detailed account, particularly of Suzuki's contributions, is given in my book [2]. If G is simple, it has been shown that either G is isomorphic to $PSL(2, q)$, $PSU(3, q)$, $Sz(2^n)$ or else $|P| = q^3$, where q is an odd power of 3, H is cyclic of order $q-1$, and $|G| = (q-1)q^3(q^3+1)$, conditions which are all satisfied by the Ree groups $R(q)$ of characteristic 3. Thompson has determined a great deal more about the structure of B and G. In particular, if B is isomorphic to a Borel subgroup of $R(q)$, he has shown that $G \cong R(q)$. However, it is at present an open question whether B must have this structure. Thus the Ree groups of characteristic 3 are the only groups of Lie type for which no recognition theorem now exists. To allow for this indeterminancy, it has been necessary, in practice, to introduce the notion of a group of Ree *type* of characteristic 3 to describe a doubly transitive group G satisfying the conditions listed above.

There exist analogous characterization theorems for each of the sporadic groups. However, in this case the nature of the theorem depends upon the way in which the given group G arose. In effect, some very precise set of internal conditions in terms of either generators and relations or subgroup structure implies that G can be realized, say, as a primitive rank 3 permutation group or as an automorphism group of the Leech lattice, at which point it is possible to recognize G.

2. Level II: centralizers of involutions. We have seen above the importance of centralizers of involutions in the discovery of new simple groups. It was undoubtedly Richard Brauer who first realized the significance of such centralizers. In fact, he proved that there exist at most a finite number of simple groups in which the centralizer of an involution has any preassigned structure. The importance of elements of order 2 in the study of simple groups was, of course, reinforced by the renowned theorem of Feit and Thompson that groups of odd order were solvable, equivalently that every nonabelian simple group has *even* order.

At the present time almost every known simple group has been characterized in terms of the structure of the centralizer of one or more of its involutions. Probably more individuals have been involved in this effort than in any other single area of simple group theory. We mention only Brauer's pioneering characterization of $PSL(3, q)$, q odd, and Suzuki's characterizations of the linear groups in characteristic 2 which together provided the impetus for the subsequent research.

I shall not attempt to describe in detail the centralizers of involutions in the known simple groups, but the situation in the linear groups typifies the structure that holds in general. It is essential here to distinguish between *odd* and *even* characteristic. In the first case, we can represent an involution x of $GL(n, q)$ as a diagonal matrix with k entries -1 followed by $n - k$ entries $+1$. Then clearly the centralizer C of x in $GL(n, q)$ consists of all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A, B are arbitrary nonsingular $k \times k$ and $(n-k) \times (n-k)$ matrices respectively. Thus

$$C \cong GL(n,q) \times GL(n-k,q).$$

Of course, we are actually interested in the centralizer of the image of x in $PSL(n,q)$, but this is easily deduced from the above structure.

When q is even, let us limit ourselves to the case that x is in the center of a Sylow 2-subgroup of $GL(n,q)$. Then x can be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \cdot \\ \cdot & 0 & 1 & & \cdot \\ \cdot & & 0 & & \cdot \\ y & \cdots & & 0 & 1 \end{bmatrix}, \quad y \neq 0;$$

and we compute that C consists of all nonsingular matrices of the form

$$\begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}.$$

In this case, we can write C as the semi-direct product $C = QK$, where

$$Q = \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & 0 & 1 & \cdots & 0 \\ * & 0 & 0 & \cdots & \\ \vdots & & & & \vdots \\ \vdots & & & \cdots & \vdots \\ * & * & * & \cdots & 1 \end{bmatrix} \right\} \text{ and } K = \left\{ \begin{bmatrix} u & 0 & \cdots & 0 & 0 \\ 0 & & & & \\ \vdots & & A & & \vdots \\ 0 & 0 & \cdots & 0 & v \end{bmatrix} \right\}$$

where A is a non-singular $(n-2) \times (n-2)$ matrix and u, v are nonzero elements of $GF(q)$. Thus Q is a 2-group, K acts faithfully on Q, and K has a normal subgroup K_0 with $K_0 \cong GL(n-2,q)$ and K/K_0 abelian. Hence in this case, C has what we may term a "unipotent radical" of characteristic 2.

In the groups of Lie type in odd characteristic, the centralizers of involutions are always essentially commuting products of groups of Lie type of lower rank, while in even characteristic, these centralizers always have a unipotent radical of characteristic 2. The alternating groups fall somewhere in between, for there the centralizers of involutions always contain a subgroup of index 2 which is the direct product of an alternating group of lower degree and another subgroup having a unipotent radical of characteristic 2. On the other hand, in the sporadic groups, with the exception of the Lyons-Sims group, the centralizer of an involution either has a unipotent radical of characteristic 2 or contains a normal subgroup of index a power of 2 which is either a group of Lie type over $GF(2^n)$ or is itself a sporadic group. (Note that the group $PSL(2,5) \cong PSL(2,4)$, so that the preceding statement is valid in Janko's group J_1, where we have previously asserted that the centralizer of every involution has the form $Z_2 \times PSL(2,5)$.) Given the nature of the centralizers of involutions in the sporadic groups, it is not unreasonable to regard them as pathologies of *groups of Lie type of characteristic* 2.

Suppose then that we specify the centralizer C of an involution x in an abstract simple group G to be that of some known simple group G^*—how do we then determine the structure of G and, if possible, show that G is isomorphic to G^*? Obviously our analysis must be aimed at showing that G satisfies precisely those conditions by which we characterized G^* at Level I. In other words, we must try to head towards a previously established recognition theorem.

I should like to outline briefly the principal steps that are used in analyzing problems of this type, for they embody general methods that are used in the study of simple groups.

a) *Fusion arguments.* The first major task is to determine the involution fusion pattern in G—that is, if S denotes a Sylow 2–subgroup of G, to give a precise description of the way in which the involutions of S are conjugate in G. We can assume S is chosen so that $S \cap C$ is a Sylow 2–subgroup of C. As is very often the case in centralizer of involution problems, x is assumed to be in the center of a Sylow 2–subgroup of G, in which case $S \subseteq C$. Since the exact structure of C has been specified, we at least know the structure of S at the outset.

The principal tools for studying involution fusion are *Glauberman's Z^*-theorem, Thompson's fusion lemma*, and *Alperin's fusion theorem.* The first asserts, under considerably weaker assumptions than simplicity, that x must be conjugate in G to some involution $y \neq x$ of S. But then $C_G(y)$ has the same structure as $C = C_G(x)$ and we can play off these centralizers against each other to obtain fusion information. We remark that Glauberman's theorem is a deep result, depending upon the theory of modular characters. By contrast, Thompson's is an easy consequence of the so-called transfer homomorphism. It asserts, under the assumption that G has no normal subgroup of index 2, that if T is a maximal subgroup of S and u is an involution of $S - T$, then u must be conjugate in G to an involution t of T. This result is used in roughly the same way as Glauberman's.

Alperin's fusion theorem asserts that the conjugacy in G of subsets of S is determined in a very precise way from the subgroups $N_G(T)$ as T ranges over the nonidentity subgroups of S for which $N_S(T)$ is a Sylow 2–subgroup of $N_G(T)$. In contrast to Glauberman's and Thompson's results, Alperin's theorem holds for all primes p. The way to think of his result is thus:

The fusion of p-elements in a group G, which is a *global* notion, is completely determined from the *local* p-structure of G.

Goldschmidt has obtained a very effective refinement of Alperin's theorem, which considerably restricts the set of subgroups $N_G(T)$ which one must examine to determine the fusion in G. Furthermore, his recent so-called strong closure theorem, which we shall describe in the chapter on general classification problems and which includes Glauberman's Z^*-theorem as a very special case, represents a powerful new tool for analyzing fusion.

b) *Centralizers of other involutions.* Once the fusion pattern of involutions is known, the next step, in the case that G has more than one conjugacy class of involutions, involves the determination of the centralizers of all, or at least some of the involutions of the remaining classes. Suppose u is such an involution. One can always take u so that x centralizes u, in which case $x \in D = C_G(u)$. But now we know the exact structure of $C_D(x) = D \cap C$. Although D is certainly not simple, it frequently happens that $\bar{D} = D/\langle u \rangle$ contains a simple normal subgroup $\bar{K}$ of small index and again we know the structure of $C_{\bar{K}}(\bar{x})$. Thus the structure of D is determined from that of $\bar{K}$. However, we see that the determination of the structure of $\bar{K}$ *involves the solution of another centralizer of an involution problem.* In practice, one would not attack a given problem concerning G unless one knew that the corresponding *subsidiary* problem for $\bar{K}$ had already been solved. This is an important point for the study of simple groups and illustrates the way in which prior classification theorems enter into the arguments.

c) *Other critical subgroup structure.* From the knowledge of the structure of the centralizers of involutions, one must now build up whatever further subgroup structure is needed to prove that G has the desired presentation. In those cases in which the goal is to demonstrate that G is of Lie type, this portion of the argument goes fairly smoothly, in general, since the (B, N)-pair structure of the groups of Lie type is in some sense not too far removed from the centralizers of their involutions. One has to construct inside of the group G appropriate candidates for B and N and prove that the subgroup G_0 which B and N generate within G is, in fact, a (B, N)-pair.

Similarly the problem is not too difficult in the case of alternating groups. However, for certain of the sporadic groups, the work involved may be very elaborate. Suppose, for example, we want G to be a primitive rank 3 permutation group in which the stabilizer of a point is to be M_{22}. Then we must prove certainly that G *has* a subgroup isomorphic to M_{22}. However, such a subgroup is not "close to the involutions", primarily because it does not occur in a local subgroup. To carry through the analysis, one needs first to determine the full character table of G, then to find an appropriate permutation representation out of this character table, and in addition, to determine elements inside of G which generate a subgroup isomorphic to M_{22}.

d) *Group order formulas.* If we return for a moment to the situation above in which we constructed a (B, N)-pair G_0 inside of G, you can see that we are still one step away from completing the proof: namely, we must show that $G = G_0$. Since we know the structure of G_0, we know its order, so we need some way of determining or at least bounding the order of G in terms of that of G_0.

This type of question arises in many situations in the study of simple groups, and essentially three distinct ways have been developed for answering it. The first of these has turned out to be one of the principal tools in the theory of simple groups, the notion of a *strongly embedded subgroup.* By the very construction of the subgroup G_0 described above, it will satisfy the following two conditions:

i. $C_G(x) \subseteq G_0$ for any involution x of G_0;

ii. $N_G(S) \subseteq G_0$ for any Sylow 2–subgroup of G_0.

In general, a proper subgroup G_0 of even order in a group G which satisfies (i) and (ii) is said to be *strongly embedded* in G.

A deep theorem of Bender gives a complete classification of groups which possess a strongly embedded subgroup; in particular, the only simple ones are the groups $PSL(2, 2^n)$, $PSU(3, 2^n)$, and $Sz(2^n)$, $n \geqq 2$. Bender's theorem is used to force the conclusion $G = G_0$, for in the contrary case, the structure of G is determined by the theorem and one then checks from the known structure of these groups that none of them contains a proper subgroup with the structure of G_0.

Since in the applications Bender's theorem allows us to draw the conclusion $G = G_0$ for some previously constructed subgroup G_0 of G, we regard it as providing a method for bounding the order of G. We note also that it is a very easy exercise to prove that in a group with a strongly embedded subgroup all the involutions are necessarily conjugate. Hence in the situation above, if we know from our fusion analysis that G has more than one conjugacy class of involutions (which is usually the case), we can conclude at once that $G = G_0$ without invoking Bender's theorem.

We remark that Bender's theorem depends in a crucial way upon Suzuki's prior classification of all (B,N)-pairs of rank 1 in which $B = HP$ and P is a 2-group. This is hardly surprising as the only such simple (B, N)-pairs are the groups $PSL(2,2^n)$, $PSU(3,2^n)$, and $Sz(2^n)$, $n \geqq 2$.

When G has more than one conjugacy class of involutions, there exists a method for computing the order of G directly from the involution fusion pattern and the structure of the centralizers of the involutions in G. It is known as the *Thompson order formula*, and its proof is remarkably elementary. For simplicity, we shall state it only in the case that G has exactly two conjugacy classes of involutions, represented by, say, x and y. First, for any involution z of G, define $a(z)$ to be the number of ordered pairs (u,v) such that u is conjugate to x and v to y in G and such that $(uv)^i = z$ for some integer i. Note that as z commutes with both u and v, $a(z)$ can be computed *within* $C_G(z)$, given a knowledge of its structure and the fusion in G of its involutions. Then Thompson's formula asserts:

$$|G| = a(y)|C_G(x)| + a(x)|C_G(y)|.$$

As the preceding discussion implicitly suggests, the analysis when G has only one conjugacy class of involutions is more difficult than when it has more than one such class. Moreover, in some of these cases the (B,N)-pair construction of the subgroup G_0 cannot be accomplished until *after* one knows the exact order of G. Indeed, this seems to be the case for the groups $PSL(2,q)$, $PSL(3,q)$, and $PSU(3,q)$, q odd, which were essentially the first groups that were considered for classification by the centralizers of their involutions. In carrying through these classification theorems Brauer, and later Suzuki, made powerful use of the ordinary and modular characters of the group G under investigation, including the theory of blocks of characters for the prime 2. Likewise the Janko groups J_1 and J_3 as well as the Lyons-Sims group, each of which arose from a centralizer of an involution problem, have only one conjugacy class of involutions and modular character theory was used to determine their orders (in the analysis which preceded the question of their existence).

Because character theory played such a dominant role in the early classification theorems, it was felt at first that this state of affairs would continue as the subject developed. However, the applications of character theory to classification problems has, in fact, been primarily limited to "small" groups—groups of Lie rank 1 and 2, for example. Gradually, as we have been considering broader and broader problems, it has turned out that internal group-theoretic techniques have sufficed for the analysis.

Before concluding this discussion of centralizer of involution problems, we wish to mention a slight extension which it has been necessary to consider at various times. Just as recognition theorems are essential for the solution of centralizer of involution problems, so classification theorems in terms of centralizers of involutions form the basis for the solution of Level III problems—characterizations of simple groups G by the structure of their Sylow 2–subgroups. However, it frequently happens, in practice, that the analysis does not quite yield that the centralizer $C = C_G(x)$ of the involution x of G is isomorphic to the corresponding centralizer C^* of some known simple group G^*, but only that C *approximates* the structure of C^*. A typical situation occurs in the case that C possesses a normal subgroup C_0 of *odd* index with C_0 isomorphic to C^*. In effect, what the analysis of G demonstrates is that C itself is isomorphic to the centralizer $\tilde{C}^*$ of an involution in some *extension* $\tilde{G}^*$ of G^* by a group of outer automorphisms of G^* of odd order. What one requires then for the application is a characterization of $\tilde{G}^*$ by the structure of its centralizer $\tilde{C}^*$. Both Harris and David Mason have considered several problems of this type, the arguments in each case being conceptually the same as those of ordinary centralizer of involution problems. At the very last line of the proof, after one has argued that $G \cong \tilde{G}^*$, one uses the initial assumption that that G is simple to conclude that $\tilde{G}^* = G^*$. Other variations of this situation have been treated and we shall refer to this whole area as *extended centralizer of involution* problems.

3. Level III: Sylow 2-subgroups. As we shall see, the solution of general classification problems require not only characterizations of simple groups by centralizers of involutions, but also by the structure of their Sylow 2–subgroups.

Characterizations of the latter type have occurred primarily when the group G under investigation has low 2–rank.

At the present time, every known simple group of 2–rank at most 4 has been characterized by its Sylow 2-subgroup. The effort that has gone into this task has been enormous. Alperin, Bender, Brauer, Gorenstein, Harada, Lyons, Mason, O'Nan, Ronald Solomon, and Walter have all had a hand in it. (We note that a group of 2-rank 1 has cyclic or generalized quaternion Sylow 2-subgroups and so, by known results, is not simple.) Partly this has been due to the fact that the 2-rank 2 cases have required very long and involved arguments (in these cases a Sylow 2-group is either dihedral, quasi-hedral, a wreath product of Z_{2^n} and Z_2, $n \geqq 2$, or isomorphic to a Sylow 2–subgroup of $PSU(3,4)$); and partly to the fact that the list of simple groups of 2–ranks 3 and 4 includes several families of groups of Lie type of odd characteristic (and low Lie rank) as well as a number of specific groups of Lie type over $GF(2^n)$ (of both low Lie rank and small values of n), the alternating groups A_n, $n \leqq 11$, and several sporadic groups. Since the structures of the Sylow 2–groups in the various cases differ markedly, a large number of independent analyses were therefore required.

The groups $PSL(2,2^n)$, $PSL(3,2^n)$, $PSU(3,2^n)$, and $Sz(2^n)$, whose 2–ranks increase with n, have also been characterized in terms of their Sylow 2–subgroups by Collins and others. Fortunately, however, there are good reasons for believing that, in general, characterizations of simple groups of 2–rank exceeding 4 by their Sylow 2–subgroups, although certainly useful, will not be essential for the solutions of general classification problems, but only characterizations by centralizers of their involutions (or, to be more precise, by extended centralizers of involutions) will be needed. I shall discuss this point in Part III, but now I would like to explain briefly what is involved in Sylow 2–group classification theorems. Up to a certain point, the argument follows the same general pattern as that in centralizer-of-an-involution problems. Again the first step is to determine the involution fusion pattern of G, using the same methods as before. However, the problem is now more elaborate, since in the former cases, a good deal of involution fusion was already "built into" the given centralizer $C = C_G(x)$.

The next step is one that was not necessary before since the structure of C was given to us in advance. But now we must use the fusion pattern in G to derive information about the structure of C. To describe this, let us define $O(X)$ for any group X to be the unique largest normal subgroup of X of *odd* order. Setting

$\bar{C} = C/O(C)\langle x\rangle$, our aim is to determine the structure of $\bar{C}$. The involution fusion pattern in G gives us information about the fusion pattern in C and hence in $\bar{C}$—and we are able to show that $\bar{C}$ possesses a normal subgroup $\bar{K}$ with $\bar{C}/\bar{K}$ abelian, or at worst solvable, and with $\bar{K}$ being "close" to a simple group if not actually simple. The essential point is that the determination of the structure of $\bar{K}$ is now reduced to a characterization of $\bar{K}$ by its Sylow 2-group and thus to a "smaller" problem of the same type as that for G.

The upshot of this analysis is that we are eventually able to determine the structure of the group $C/O(C)$ to the following extent: we can find an extension $\tilde{G}^*$ of a known simple group G^* with $\tilde{G}^*$ and G^* having Sylow 2-subgroups isomorphic to those of G such that if $\tilde{C}^* = C_{\tilde{G}}*(x^*)$, where x^* is an appropriate involution of G^*, then

$$C/O(C) \cong \tilde{C}^*/O(\tilde{C}^*).$$

Thus we can see exactly what remains to be done to reduce our desired classification theorem to a prior extended centralizer-of-an-involution theorem—namely, we must show that $O(C)$ and $O(C^*)$ have "essentially" the same structure.

This task represents a major step and in some cases the bulk of the analysis. When the structure of $O(C)$ is more complex than that of $O(\tilde{C}^*)$, the goal, in general, is to show that a maximal subgroup of G containing C is strongly embedded in G and then to invoke Bender's theorem to derive a contradiction.

In carrying this out, the 2-rank 2 cases differ sharply from those in which G has 2-rank 3 or 4. Indeed, in the latter cases there exists a uniform technique for accomplishing it which is rapidly becoming a fundamental, general method for the study of simple groups and which I shall describe in a moment. On the other hand, in the 2-rank 2 cases, the subgroup structure of G related to suitable *odd* primes must be brought into play in an essential way, using ideas that were first introduced in the Feit-Thompson work on groups of odd order. Moreover, there turn out to be several residual cases where these fairly involved methods themselves break down and one must resort to character theory and block theory to complete the analysis. Some discussion in the special case of groups with dihedral Sylow 2-subgroups is given in [2, Chap. 16].

In discussing 2-rank 3 or 4 problems, let us limit ourselves to the case in which for any involution y^* of $\tilde{G}^*$, we have $O(C_{\tilde{G}}*(y^*)) = 1$, in which case our object will clearly be to prove that $O(C_w(y)) = 1$ for any involution y of G; and, in particular, that $O(C) = 1$. In this situation the problem reduces to showing that G is what is called a *balanced* group; that is,

For any pair of commuting involutions u, v of G, we have

$$O(C_G(u)) \cap C_G(v) = O(C_G(v)) \cap C_G(u).$$

To establish this property of G, we need to know the structure of $C_G(y)/O(C_G(y))$ for involutions y of G (if any) that are not conjugate to x. This is an analogous situation to one that we considered in centralizer-of-involution problems and, in general, it is handled in the same way. However, in some cases because of the presence of the group $O(C)$, over whose structure we have no control at the outset, one is forced to carry through an analysis of the p-structure of G for some single odd prime p which is reminiscent of certain arguments of the 2–rank 2 cases.

Once we know that G is balanced, the proof proceeds in a very smooth way, for we can invoke the basic so-called *signalizer functor theorem*. My original proof of this theorem was fairly involved and represented an attempt to put certain arguments of the odd order paper into a general setting (essentially the same ones to which we referred above). That portion of the odd odd order paper was subsequently simplified by Bender, and his ideas have had a significant influence. In particular, Goldschmidt applied them to make sharp improvements in both the statement and proof of the signalizer functor theorem, so that it is now very easy to use in the study of simple groups. Moreover, his theorem covers the 2–rank 3 case, which mine did not.

Let A be an elementary abelian 2–subgroup of G of maximal rank containing our involution x, so that A has 2–rank 3 or 4, as the case may be. The condition for balance when restricted to the involutions of A asserts simply that the mapping: $a \to O(C_G(a))$ for a in $A^{\#}$ defines an *A-signalizer functor on G*. The signalizer functor theorem then yields that the group

$$W_A = \langle O(C_G(a)) \mid a \text{ in } A^{\#} \rangle$$

has *odd* order.

Because of this, W_A is certainly a *proper* subgroup of G. Hence either $W_A = 1$ or $N_G(W_A)$ is also a proper subgroup of G as G is simple. However, in the latter case it is not difficult to prove in the given problems that $N_G(W_A)$ is, in fact, strongly embedded in G, and now Bender's theorem yields a contradiction. Thus $W_A = 1$. Since $x \in A$, we therefore conclude that $O(C_G(x)) = O(C) = 1$, thus completing the proof that $O(C)$ is the same as $O(\tilde{C}^*)$.

In those cases in which $O(C_{\tilde{G}^*}(y^*)) \neq 1$ for some involution y^* of G^*, the proof follows a similar line, but the construction of an appropriate A-signalizer functor on G is somewhat more complicated and involves the theory of *k-balanced* groups, primarily for $k = 2$. (1–balanced groups are the same as balanced groups.) k-balanced groups are discussed in some detail in the Oxford lectures.

Finally a word about the group $\bar{K}$ above which, as we have noted, need not be simple. In many problems, $\bar{K}$ should turn out to be a direct product of two (or more) simple groups. For example, in the characterization of $PSp(4, q)$, q odd, by its Sylow 2–group, a Sylow 2–subgroup of $\bar{K}$ will be the direct product of two dihedral groups and our goal must be to prove that $\bar{K} \cong PSL(2, q) \times PSL(2, q)$. To obtain the desired conclusion, it is necessary to have a classification of all groups with Sylow 2–groups which are the direct product of two dihedral groups. Using the same general methods we are describing, Harada and I had previously established just such a classification. We see then that characterizations of simple groups by their Sylow 2–subgroups often require subsidiary results which, in effect, constitute characterizations of direct products of simple groups rather than of simple groups themselves.

4. Other classification problems. A considerable number of special classification problems concerning simple groups have been investigated besides those described above. In some instances the results obtained have already found application in general classification problems. In other cases, such applications may never occur. Quite apart from the question of their applicability, each of these problems represents a major area of research, which has had important influence on our understanding of simple groups. Moreover, at the very least, we can think of these investigations as a search for new simple groups. And so I would like to describe the major ones very briefly.

a) *Multiply transitive permutation groups.* In view of the Mathieu groups, we have seen the importance of this area. The following result of Wielandt will indicate the status of our present knowledge. To state it, I must first mention the Schreier conjecture which asserts that the outer automorphism group of every simple group is solvable. This indeed is the case for the known simple groups. Wielandt's theorem asserts that if the Schreier conjecture holds, then the only 8–fold transitive permutation groups are S_n and A_n. Using Wielandt's methods, Nagao obtained the same result under the weaker assumption of 7–fold transitivity. Using much deeper methods, the same conclusion has just been obtained by O'Nan for 6–fold transitive groups.

To date there has been very little progress on the Schreier conjecture, just a single beautiful but very special theorem of Glauberman, and it looks as though its verification will depend upon the classification of all finite simple groups. In this sense then, the determination of all 6–fold transitive permutation groups will follow from the classification of the simple groups.

Curtis, Kantor, and Seitz have determined all the known simple groups which possess a doubly transitive permutation representation. Hence, in fact, the classification of all finite simple groups would yield as a corollary all simple doubly transitive groups.

We remark that O'Nan's proof of the above result is a consequence of his recent deep work on doubly transitive groups. His results indicate that we may be close to the following general conclusion:

> If G is a doubly transitive permutation group having no regular normal subgroup, then either G is of known type or else the stabilizer of a point is a holomorph of a simple group.

We say that H is a *holomorph* of a simple group L if H is isomorphic to a subgroup of $\mathrm{Aut}(L)$ containing $\mathrm{Inn}(L)$.

In effect, this result would reduce the determination of all doubly transitive groups to the possible transitive extensions of holomorphs of simple groups. Unfortunately, it looks as though the determination of all doubly transitive groups would still require the classification of all simple groups, even with this result.

b) *Primitive rank* 3 *permutations groups, automorphism groups of integral lattices, and groups determined by a conjugacy class of involutions.* We have lumped these three distinct topics together because of their obvious connection with the sporadic groups. As I stated before, much effort has been expended in each of these directions. I think it is reasonable to regard Rudvalis' newly discovered group as the fruit of this labor[†]. Apart from this, most of the positive results have occurred in obtaining extensions of Fischer's classification theorem. In particular, Aschbacher has determined all simple groups which can be generated by a conjugacy class of involutions, the product of any two of which has order one, two, or some (variable) *odd* number. This is an important result and the answer includes many families of groups not covered by Fischer's theorem; unfortunately no new simple groups arise (apart from the Fischer groups themselves).

[†] Plus the new Fischer group mentioned in a preceding footnote.

We remark that analogous investigations have begun of a group G which can be generated by a conjugacy class of elements of prime order p, p *odd*, such that every pair of elements of the class generate a subgroup of G with some specified structure. This problem is closely related to the theory of *quadratic pairs* which we consider next.

c) *Quadratic pairs*. We have mentioned above certain lines of argument that stemmed from the odd order paper. A fundamental tool in carrying out analyses of this type, as well as in many other situations, is the basic ZJ-theorem of Glauberman that gives a sufficient condition for a Sylow p-subgroup P of a group H, p *odd*, to possess a nontrivial characteristic subgroup which is normal in H. The essential hypothesis of this theorem relates to the concept of *p-stability* for the group H, which reduces in turn to questions concerning representations of certain sections of H on vector spaces over Z_p (that is, over $GF(p)$).

Thus a group of linear transformations G acting on a vector space V over Z_p, p odd, is said to be *p-stable* if G contains no element x of order p whose minimal polynomial on V is $(X - 1)^2 = 0$. In the contrary case, G is said to be *non p-stable.* In that case G contains an element x of order p which has a *quadratic* minimal polynomial on V.

For Glauberman's theorem, the critical sections of H must have p-stable representations on the corresponding vector spaces. In general, groups of Lie type of characteristic p have non p-stable representations (in fact, all except E_8). For example, in the natural representation of $GL(n, p^m)$ an element x with ones on the diagonal and only one non-zero entry off the diagonal has $(X - 1)^2 = 0$ as its minimal polynomial. Such an element x is called a *transvection* and the phenomenon of non p-stability is closely related to the transvections in the given group G.

The case of prime interest occurs when the linear group G acts irreducibly and non p-stably on V, and G is generated by the conjugates of the element x. In this case, the pair (G, V) is called a *quadratic pair.*

In a remarkable piece of work Thompson has classified all quadratic pairs when $p \geqq 5$ and has shown that G must be a group of Lie type of characteristic p. A major step in his analysis is the determination of the possible subgroups of G generated by two elements of the conjugacy class of x. This shows the intimate connection between quadratic pairs and the "Fischer problem".

The difficulty in the case $p = 3$ is caused, in part, by the fact that there are more possibilities for the list of subgroups which two conjugates of x can generate. We note that this case is of especial interest because the largest Conway group is, in fact, a quadratic pair for the prime 3.

d) *Groups with a strongly embedded subgroup.* Earlier we described Bender's theorem. An important extension of this result has been obtained by Aschbacher. Refining Bender's argument, he has classified all groups which have a proper 2–generated core. By definition, the *k-generated core* of a group G is the subgroup

$$\langle N_G(T) \mid T \subseteq S, \text{ 2–rank of } T \text{ is at least } k \rangle,$$

where S is a fixed Sylow 2–subgroup of G. Clearly the k-generated core is determined up to conjugacy by the Sylow 2–subgroup S of G. It is easy to see that a group G of even order has a proper 1–generated core if and only if its 1–generated core is strongly embedded in G. Thus Aschbacher's theorem is a true extension of Bender's result. In particular, Aschbacher has shown that a simple group with a proper 2–generated core either has a strongly embedded subgroup (and hence is isomorphic to $PSL(2,2^n)$, $Sz(2^n)$, or $PSU(3,2^n)$, $n \geqq 2$) or else is isomorphic to Janko's first group J_1.

We remark that a major portion of Bender's proof consists in the classification of doubly transitive permutation groups in which the 2–point stabilizer has odd order (corresponding to the special case in which the permutation representation on the right cosets of the strongly embedded subgroup is doubly transitive). It is this portion of the work which utilizes Suzuki's deep results on (B,N)-pairs of rank 1.

e) *Groups with a (B,N)-pair.* We shall not attempt to add to our earlier comments concerning (B,N)-pairs and their classification. However, we do wish to list this important topic, which has had very considerable impact on the study of simple groups.

f) *Groups with a low degree representation.* Brauer and several of his students have studied finite groups which possess a complex representation of low degree, primarily up to 7 and the determination of such groups is largely complete. Perhaps the only surprises have come from Lindsey who has shown that the 2–fold covering group of J_2 has a complex representation of degree 6, while the sporadic group of Suzuki possesses a central extension by Z_6 which has a complex representation of degree 12. (Note that the order of this latter group is approximately 1.5 trillion $(1.5)10^{12}$.)

Using modular character theory, Brauer, and later Feit as well as several others, have investigated groups which are divisible by an odd prime p only to the first power and which possess a complex representation of low degree n relative to p, primarily n is the range $p/2 \leqq n \leqq 3p/2$. Again, unfortunately, only known groups have arisen.

g) *Arithmetical problems.* Many of the sporadic groups have been characterized solely in terms of their group order. Another problem area has been the study of simple groups G of order $p^a q^b r$, p, q, r distinct primes. By Burnside's theorem, all groups of order $p^a q^b$ are solvable, so this was a natural first case to consider. By the Feit-Thompson theorem together with Thompson's classification of the so-called minimal simple groups (both of which we shall discuss when we some to general classification problems), it suffices to consider the cases that

$$|G| = 2^a \cdot 3^b \cdot r, \text{ where } r = 5, 7, 13, \text{ or } 17.$$

By a combination of group theory and modular character theory, Brauer treated the case $r = 5$ and subsequently Wales handled the cases $r = 7, 13$, and 17. Their combined work shows that there exist exactly eight such simple groups, all well-known. Recently their results have been generalized to groups of order $2^a 3^b r^c$ in which a Sylow r-subgroup is cyclic†.

h) 3–*structure.* Evidence has been slowly emerging from the various classification theorems treated to date that the 3–structure of simple groups is of basic importance, second only to that of its 2–structure. I believe that in the near future this will become one of the major areas of research, primarily in obtaining characterizations of groups of Lie type of *even* characteristic and certain sporadic groups in terms of the structure of the centralizers of elements of order 3, analogous to the characterizations by centralizers of involutions we have described above.

At the present time only scattered results exist. Thus Feit and Thompson have determined all simple groups which contain a self-centralizing subgroup of order 3. Graham Higman and several of his students have obtained what can be regarded as extensions of the Feit-Thompson theorem. They have also investigated, by analogy with the original centalizer of involution problem of Suzuki, groups in which the centralizer of every element of order 3 is a 3–group. Recently a student of mine has obtained a partial characterization of $PSL(4, 2^{2n})$ in terms of the structure of the centralizer of an element of order 3 and Fischer has obtained similar results for certain of the sporadic groups. Finally in a major general classification theorem, Thompson has classified all simple groups of order relative prime to 3, the Suzuki groups $Sz(2^n)$ being the only such groups.

† Klinger and Geoffrey Mason have recently shown that PSL (3,3) is the only simple group of order $2^a 3^b 13^c$.

i) *Fusion of p-elements.* Glauberman has been relentlessly analyzing the fusion of p-elements in an arbitrary group G. By very elaborate refinement of the methods of his ZJ-theorem, he has shown that for $p \geqq 5$ there exists a *single p-local* subgroup H of G which controls p-fusion in G. This means that H contains a Sylow p-subgroup P of G and two elements of P are conjugate in G if and only if they are conjugate in H. This profound result implies as a corollary a long-standing question in finite group theory:

There exists no simple group in which every Sylow subgroup of the group is self-normalizing.

5. Properties of the known simple groups. As we have indicated, the study of the internal structure of a simple group G depends upon the nature of the simple groups involved in the proper subgroups of G and, in practice, the critical such groups are of known type. As a result, essentially all classification theorems involve, in a crucial way, specific properties of known simple groups. Thus the substantiation of such properties becomes an integral part of the theory of simple groups and although this is not strictly speaking a type of classification problem, it certainly deserves the same status and recognition as any of the problem areas that we have considered above.

We shall limit ourselves here to presenting a list of the various types of properties of known simple groups K that seem to be needed for the applications.

a) Centralizers of involutions and, more generally, of elements of prime order in K.

b) Local structure.

c) Automorphisms.

d) Schur multipliers, that is, the determination of the unique largest perfect central extension of K by an (abelian) group A.

e) Generation. The following is typical: if P is an elementary abelian p-subgroup of K (or of Aut K), p a prime, when is K generated by its subgroups $C_K(x)$ as x ranges over $P^\#$?

f) Irreducible representations on vector spaces over $GF(p)$, primarily, but not exclusively, when K is of Lie type of characteristic p.

g) Ordinary and modular complex representations, primarily, but again not exclusively, in the case that K is "small".

III. Level IV. General Classification Problems

We repeat our definition of a *general classification* problem as one in which the assumed property X is inductive to all subgroups and homomorphic images. Proceding by induction on the order, it is immediate that a minimal counter-example G is then simple and the (nonsolvable) composition factors of each of its proper subgroups are known simple groups having property X. In many ways, it gives a clearer picture of the situation to consider this to be the definition of a general classification problem.

Thus a *general classification theorem* amounts precisely to the determination of all simple groups G in which the composition factors of every proper subgroup of G are members of some predetermined family $\mathscr{F}$ of known simple groups.

In general, the goal is then to prove that G itself is an element of $\mathscr{F}$. However, this is not essential; the basic consideration is to determine the possibilities for G. Moreover, in practice, the analysis does not require the given assumption on the composition factors of *all* proper subgroups of G, but only on all *local* subgroups.

As we mentioned before, some of the problems we treated as special can be considered to be general classification problems. This is particularly true of Sylow 2–subgroup characterization theorems. For example, the classification of groups with dihedral Sylow 2–subgroups leads quite easily to a general classification problem in which the family $\mathscr{F}$ consists of the groups $PSL(2,q)$, q odd, and A_7, these being the only known simple groups with dihedral Sylow 2–subgroups.

Apart from those special classification problems which can be considered to be general ones, there exist at the present time exactly seven problems which have been successfully treated and which we list in chronological order of their completion†.

a) Groups of odd order.

b) Groups with abelian Sylow 2–subgroups.

c) Groups each of whose local subgroups is solvable (for brevity, N-groups). In particular, *minimal* simple groups—simple groups all of whose proper subgroups are solvable.

d) Thin groups with solvable 2–local subgroups (a group is called *thin* if each of its 2–local subgroups has cyclic Sylow subgroups for all odd primes).

e) Groups each of whose 2–subgroups can be generated by at most four elements (for brevity, groups of *sectional* 2–rank at most 4).

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† To this list, we may now add: groups whose 2-local subgroups are solvable (Gorenstein-Lyons-Smith) and groups of order $2^a3^b13^c$.

f) Groups of order relatively prime to 3.

g) Groups with Sylow 2–subgroups of nilpotency class 2.

1. Groups of odd order. A detailed outline of the Feit-Thompson proof that all groups of odd order are solvable is given in [2, Chap. 16] and I shall not attempt to repeat any of it here; instead I shall limit myself to a few comments. In the dozen or so years since this fundamental paper appeared, it has had a profound influence on the study of simple groups. In particular, two of the basic techniques used in general classification problems, the signalizer functor and Bender methods, stem from the uniqueness theorems of Chapt. IV. Moreover, the factorization lemmas used in deriving these uniqueness theorems led, on the one hand, to a very simple proof of Thompson's celebrated proof of the Frobenius conjecture that a group admitting a fixed-point-free automorphism of prime period must be nilpotent and, on the other hand, provided one of the major tools in Thompson's work on N-groups and $3'$-groups.

All the marvelous techniques of that paper have now been fairly well assimilated—the Hall-Higman theorem leading to the notion of p-stability, the factorization lemmas suggesting the ZJ-theorem, the importance of the uniqueness theorems for group theory, Dade's conceptualization of the critical character ring isometries of Chapts. III and V. And if this understanding were brought to bear on the original 255 page proof, perhaps it could now be reduced to 150 pages. However, and what is more significant, the essential nature of the proof would remain unchanged—no conceptual simplications have been made in the original argument.

2. The basic subdivision. Apart from groups of odd order, the general classification theorems, listed above, will undoubtedly provide the basis for all major undertakings in the study of simple groups in the years ahead and so it is very important to understand as fully as possible the conceptual nature of their proofs; in particular, to what extent the lines of argument suggest ways of attacking broader and broader general classification problems. We should therefore first like to analyze their proofs in general terms.

For this purpose, it is best to divide the theorems into two types:

1) Small phenomena: groups of 2–rank at most 4, thin groups.

2) Large phenomena: N-groups, $3'$-groups, groups with abelian and class 2 Sylow 2–subgroups.

A very brief outline of Thompson's monumental classification of N-groups also appears in my book. That outline stresses the fact that the over-all proof breaks up into major parts, one portion constituting the more-or-less general case, and the remainder divided into two special cases, both related to small phenomena, but one involving the prime 2 and the other involving odd primes in connection with the 2–structure. A similar overview of the classification of groups of order relatively prime to 3 reveals an analogous subdivision, but with one difference, that the general case itself splits into two parts, one entirely similar to that of the N-group analysis, and the other corresponding to a new situation that did not appear at all in the N-group argument. Moreover, in the abelian and class 2 Sylow 2–group theorems, it is this second situation which constitutes the bulk of the analysis. Again there are also small cases which must be considered separately.

Thus the overall proofs of these theorems seem to divide into *four* major parts: two connected with small phenomena and two with a fundamental subdivision of the general case. For the present let us just say that this division of the general case corresponds in a natural way to differences that we have noted earlier between the structure of the centralizers of involutions in the groups of Lie type, on the one hand, of *even* characteristic and, on the other, of *odd* characteristic.

The discussion so far has been intentionally vague and we wish next to formalize this subdivision somewhat more precisely. However, we should first remark that in the four general classification theorems listed in (2) there is a considerable difference in the proportion of the proof represented by each of the four parts of this subdivision. Despite this fact, there is sufficient evidence now that broader general classification problems will also involve the same major subdivisions in their analysis.

We begin with some definitions that are rapidly becoming standard terminology. A group L is called *quasisimple* if it is a perfect central extension of a simple group (that is, $L/Z(L)$ is simple and $[L, L] = L$). Furthermore, a group L is said to be *semisimple* if L is a central product of quasisimple groups (that is, $L = L_1 L_2 \cdots L_r$ with L_i quasisimple and $[L_i, L_j] = 1$ for $i \neq j$). The L_i are called the *components* of L and are uniquely determined by L itself. It is also convenient to call the trivial group of order 1 *semisimple.*

The importance of this notion rests on the fact that every group X possesses a unique maximal normal semisimple subgroup. We denote it by $L(X)$ and call it the *layer* of X. In practice, the components of $L(X)$ will be central extensions of known simple groups, which will explain why we wish to know the Schur multipliers of every known simple group†.

† In fact, these are now all known, most of the open cases having been treated by Griess in his doctoral thesis.

For any group X and any prime p, $O_{p'}(X)$ and $O_p(X)$ denote, respectively, the unique largest normal subgroup of X of order relatively prime to p and of order a power of p. In particular, $O(X)$ and $O_{2'}(X)$ are the same subgroup of X.

If $O_{p'}(X) = 1$, we say that X is *p-constrained* provided $C_X(O_p(X)) \subseteq O_p(X)$. Thus $O_p(X)$ contains its own centralizer in X. More generally, we say that X is p-constrained if $X/O_{p'}(X)$ is p-constrained. (Note that $O_{p'}(X/O_{p'}(X)) = 1$.) This concept generalizes a basic property of solvable groups, for a solvable group is p-constrained for every prime p.

There is a close connection between p-constraint and the layer of X. Indeed, if $O_{p'}(X) = 1$, it is easy to prove that X is p-constrained if and only if the layer of X is trivial.

These two general concepts are very important for understanding the general structure of the centralizers of involutions in simple groups. To see this, take X to be the centralizer C of the involution x of $GL(n,q)$ which we considered in the preceding part.

If q is odd, $X \cong GL(k,q) \times GL(n-k,q)$. But now we check that for most values of n and q, we have $L(X) \cong SL(k,q) \times SL(n-k,q)$ if $k \neq 1$ or $n-1$ and $L(X) \cong SL(n-1,q)$ in the contrary case. Moreover, $X/L(X)$ is abelian. Hence in this case, the layer of X dominates the structure of X.

On the other hand, if q is even, we have that $X = QK$, where Q is what we previously called the unipotent radical of characteristic 2 and K acts faithfully on Q. These conditions imply that $O(X) = 1$ and that $C_X(Q) \subseteq Q$. Moreover, by the structure of K, it has no nontrivial normal 2–subgroups and consequently $Q = O_2(X)$. Hence in this case, we see that X is 2–constrained and so it is the unipotent radical of characteristic 2 which dominates the structure of X.

It is this dichotomy that holds in all the groups of Lie type, which accounts for the break-up of the general case in a classification problem into two major subcases.

To differentiate these two possibilities, in general, we introduce the following terminology. We shall say that an arbitrary simple group G is of

Component type if for some involution x of G, $C_G(x)/O(C_G(x))$ has a nontrivial layer; and is of

Non-component type if for every involution x of G, $C_G(x)/O(C_G(x))$ has a trivial layer and hence is 2–constrained.

Simple groups of non-component type are essentially the same as what I have once termed a group of *characteristic* 2 *type*. Indeed, there is a general proposition which asserts that if the centralizer of every involution of a group G is 2–constrained, then G is, in fact, balanced. John Walter and I have studied balanced groups in some detail. A particular case of our analysis yields quite easily that if the centralizer of every involution of G is 2–constrained and if a Sylow 2–subgroup of G is not "too small", then $O(C_G(x))=1$ for every involution x of G. Actually if one combines our analysis with Aschbacher's classification of groups with a proper 2–generated core, one can reach the same conclusion for arbitrary balanced groups (again assuming Sylow 2–subgroups are not too small). In the 2–constrained case, one can deduce further that every 2–local subgroup H of G is 2–constrained and satisfies $O(H) \neq 1$. These latter conditions constitute the definition of a group of characteristic 2 type. Thus a simple group of non-component type with not too small a Sylow 2–subgroup is necessarily of characteristic 2 type, which is an important preliminary fact for its subsequent analysis.

We need one final term. The *2–local p-rank* of G, p an odd prime, is by definition the maximum p-rank of some 2–local subgroup of G. Thus a thin group is simply a group whose 2–local p-rank is at most 1 for all odd primes p.

We can now describe the four basic pieces of the analysis of a general classification problem. Although the existing proofs have not necessarily followed the order in which we shall list these parts, we feel that the one we have chosen is the best one for visualizing the nature of the proof:

A) Groups of low 2–rank;

B) Groups of component type;

C) Groups of non-component type of low 2-local p-rank, p odd;

D) Groups of non-component type.

We have deliberately not specified the exact meaning of "low". In the problems so far considered, low 2–rank has meant 2–rank at most 4 and low 2–local p-rank has meant 2–local p-rank at most 2 for all odd p. We have also not attempted to make the subdivision mutually exclusive. Thus, in practice, the study of groups of low 2–rank has included groups of both component and non-component type.

Finally we emphasize the fact that, in general, the groups of Lie type of odd characteristic, the alternating groups, and certain sporadic groups are of component type, while the groups of Lie type of even characteristic and the remaining sporadic groups are of non-component type.

3. Some general observations. I should like to make a few remarks now about the lengths of the proofs in finite group theory, particularly in general classification therorems. Thompson has described the simple groups as *extremal* among the set of all finite groups and it is this point that I wish to illuminate.

Consider, for example, groups with abelian Sylow 2–subgroups. In this case the family $\mathscr{F}$ consists of the groups $PSL(2,q)$, $q \equiv 3,5 (\mathrm{mod}\ 8)$, $PSL(2,2^n)$, J_1, and the groups of Ree type of characteristic 3. Moreover, if G^* denotes an arbitrary group with abelian Sylow 2–subgroups whose nonsolvable composition factors are in $\mathscr{F}$, then G^* has the following structure. If we set $\bar{G}^* = G^*/O(G^*)$, then $\bar{G}^*$ contains a normal subgroup $\bar{L}$ such that

a) $\bar{L} = \bar{L}_0 \times \bar{L}_1 \times \cdots \times \bar{L}_r$, where $\bar{L}_0$ is a 2–group and $\bar{L}_i \in \mathscr{F}$, $1 \leqq i \leqq r$; and
b) $\bar{G}^*/\bar{L}$ has odd order.

This then is the general structure of a known group with abelian Sylow 2–subgroups.

Now consider the general classification problem for groups with abelian Sylow 2–subgroups. Proceeding by induction on the order of the given group, one is reduced to considering a simple group G all of whose proper subgroups are of known type. The way to picture G at the *outset* is the following: the *proper* subgroup structure of G is as complicated as that of the general group G^* described above. This means that even though G is simple, we imagine that G is isomorphic to G^* in order to get some feeling for the nature of the proper subgroup structure of G at the beginning of our analysis. In other words, G^* provides a "prototype" foɹ the internal structure of G. Thus, if x is an involution of G, the structure of $C = C_G(x)$ can be as involved as the structure of $C_{G^*}(x^*)$ for an involution x^* of G^*. In particular, we see that $O(C)$ can have as complex a structure as we wish and $L(C/O(C))$ can have arbitrarily many components.

On the other hand, the object of the proof is to demonstrate that G itself is isomorphic to an element F^* of $\mathscr{F}$, since this is what is required to complete the induction argument. In particular, the internal structure of G must be shown to resemble that of F^* rather than that of the arbitrary group G^*. However, the internal structure of F^* is extremely restricted compared to that of G^*. For example, the centralizer of an involution in F^* is, according to the various possibilities for F^*, either a dihedral group, a 2–group, or is isomorphic to $Z_2 \times PSL(2,q)$, where $q = 5$ or 3^m, m odd, $m \geqq 3$.

Similarly if one considers the problem of minimal simple groups G (or, more generally, N-groups), the prototype G^* for G is either the most general solvable group or a subgroup of the automorphism group of one of the known simple N-groups. On the other hand, the internal structure of an element F^* of the corresponding family $\mathscr{F}$ is again extremely limited. For example, in almost all cases the Syow p-subgroups of F^* are abelian (and usually cyclic) for every odd prime p.

Each general classification problem proceeds in this way by induction and one reduces the problem to consideration of a simple group G with the property that the nonsolvable composition factors of the proper subgroups of G lie in the corresponding family $\mathscr{F}$ of known simple groups—that is, satisfy the conclusion of the theorem in question. The goal of the analysis is then to force the internal structure of G, which at the outset appears quite arbitrary, to have a very tight form: namely, to resemble that of one of the elements of $\mathscr{F}$. The simplicity of G enters into this analysis in precisely one way: it enables us to deduce that certain subgroups which we construct inside of G are necessarily proper, which in turn lead to contradictions. These contradictions have the effect of excluding certain configurations of subgroups from existing in G. In general, we require a long succession of such constructions and contradictions to mold the internal structure of G into its final shape. We may compare the entire process to that of a sculptor who slowly and painstakingly chisels a block of marble into an elegant, finished form.

The basic four-part subdivision described in the last section constitutes simply the first crude blocking out of the problem. Within each part, further case subdivisions are often required. For example, when G is of component type, one distinguishes subcases according to the nature of the components of $L(C_G(x))/O(C_G(x))$—x an involution of G—in particular, whether or not some such component is of Lie type of odd characteristic. It will be convenient if such is the case, to say that G is of *odd* component type.

The whole thrust of the analysis, in every subcase that must be considered, is to head towards a previously established Level II or Level III characterization theorem for those elements of $\mathscr{F}$ in the given problem which fall into that particular subcase. In general, that is, in those subcases in which G is not "small", only Level II characterizations seem to be needed (which will explain our earlier comment that Sylow 2–group characterization theorems for simple groups of large 2–rank are not essential for general classification theorems).

It should be clear from this description that to reach the extremal internal structures that characterize the known simple groups by the methods we have developed, the proofs must inevitably be very long. As Thompson has also said, we are either being very stupid in our approach to the finite simple groups or else very clever indeed.

4. Groups of low 2-rank. In Sections 3–6 I should like to describe the proofs of the six listed general classification theorems in relation to the basic four-part subdivision. The study of simple groups of 2–rank at most 4 has involved perhaps the most sustained effort of any single area of group-theoretic research and at the present time is hopefully near its completion.

In the case of simple groups G of 2–rank 2, almost all the work occurs in the Sylow 2–group characterizations, for Alperin has shown — by quite a short argument based primarily on involution fusion analysis, but depending also on a structure result for 2–groups admitting an automorphism of odd order with special properties—that a Sylow 2–subgroup of G is either dihedral, quasi-dihedral, wreathed, or is isomorphic to a Sylow 2–subgroup of $PSU(3,4)$. His result, combined with the previously established classification theorems, thus yields a complete determination of all simple groups of 2–rank 2. They are the groups $PSL(2,q)$, $PSL(3,q)$, $PSU(3,q)$, q odd, A_7, M_{11}, and $PSU(3,4)$. (Note that $A_5 \cong PSL(2,5)$ and $A_6 \cong PSL(2,9)$.)

Hence we can focus our attention on the 2–ranks 3 and 4 cases. In his work on N-groups G, Thompson found that for technical reasons connected with transitivity theorems, he was forced to treat the case that a Sylow 2–subgroup S of G did not contain an abelian *normal* subgroup of rank 3. In his terminology, we write that $SCN_3(S)$ (or simply $SCN_3(2)$) is empty. We remark that the corresponding situation in the odd order paper had also been exceptional for analogous reasons.

It therefore became important to determine all simple groups G in which $SCN_3(2)$ was empty. Thompson accomplished this under the assumption that G was an N-group. Subsequently he and Janko, jointly, extended his results to the case that only the centralizer of every involution of G was assumed to be solvable. Following this, MacWilliams studied the general situation and completed the analysis in the special case that a Sylow 2–subgroup S of G contains a normal subgroup $U \cong Z_2 \times Z_2$ whose three involutions are conjugate in G. (The existence of a normal $U \cong Z_2 \times Z_2$ in S follows, in any event, if we assume G has 2–rank at least 3.) In each of these investigations the goal was to force the structure of S by the same methods as in the 2–rank 2 case (combined with the additional assumptions on the subgroup structure of G in the Thompson-Janko situations). I should add that Alperin's work came after and to some extent was inspired by these $SCN_3(2)$ results.

Also, as I remarked earlier, one of the 2–groups which MacWilliams listed turned out to be the Sylow 2–subgroup of the Lyons-Sims group. Her list also included a certain infinite family of 2–groups, each member of which contained the group Z_{2^n} wreath Z_2 as a subgroup of index 2 for some n. By a more detailed fusion analysis, Harada was able to show that none of these groups could occur as the Sylow 2–subgroup of a simple group. This had the effect of reducing her list to the Sylow 2–subgroups of Janko's group J_2, the Lyons-Sims group, and the groups $G_2(q)$, $q \equiv 1, 7 \pmod 8$ (assuming the 2–rank of G is at least 3). Since each of these groups has a Level III characterization in terms of the structure of its Sylow 2–subgroup, these results together with Harada's led to a complete determination of all simple groups which satisfy MacWilliams's initial hypothesis.

An important subsidiary result of MacWilliams was the following property of a 2–group S in which $SCN_3(S)$ is empty:

Every subgroup of S can be generated by at most 4 elements.

Thus if G has $SCN_3(2)$ empty, then the sectional 2–rank of G is at most 4. However, the first condition is not inductive to subgroups or homomorphic images, while the second one is.

This suggested that it might be better to approach the problem of determining all simple groups G with $SCN_3(2)$ empty as a part of the general classification problem of groups of sectional 2–rank at most 4. Harada and I undertook this task two years ago and we have recently completed the analysis. Aside from groups of 2–rank 2 and groups with abelian Sylow 2–subgroups (of 2–rank at most 4), our theorem asserts that the only such simple groups are $G_2(q)$, $D_4^2\ (q)$, (the so-called triality Steinberg group obtained from the Chevalley group $D_4(q)$), $PSp(4, q)$, q odd, $PSL(4, q)$, $q \not\equiv 1 \pmod 8$, $PSU(4, q)$, $q \not\equiv -1 \pmod 8$, $PSL(5, q)$, $q \equiv -1 \pmod 4$, $PSU(5, q)$, $q \equiv 1 \pmod 4$, $PSL(3, 4)$, $Sz(8)$, A_n, $8 \leqq n \leqq 11$, $M_{12}, M_{22}, M_{23}, J_2, J_3$, McLaughlin's group, and the Lyons-Sims group. As a corollary, we obtain the complete list of all simple groups G in which $SCN_3(2)$ is empty.

Again the object of the analysis is to pin down the possible structures of a Sylow 2–subgroup S of G. However, the full proof is extremely long (over 400 pages) and itself divides into *three* major cases. To describe these, we introduce the term *central* for an involution which lies in the center of some Sylow 2–subgroup of G, and *noncentral* for an involution which lies in the center of no Sylow 2–subgroup of G. Then the breakup is as follows.

1) *The 2–constrained case.* One of the following holds:

a) Every 2–local subgroup of G is solvable (and hence 2–constrained);

b) The centralizer of some involution is nonsolvable and 2–constrained;

c) The centralizer of every central involution is solvable and some 2–local subgroup is nonsolvable and 2–constrained.

2) *The central case.* The centralizer of some central involution of G is not 2–constrained (and hence has a nontrivial component).

3) *The noncentral case.* The centralizer of every central involution of G is solvable, every nonsolvable 2–local subgroup of G is not 2–constrained, and the centralizer of some noncentral involution is nonsolvable (and hence not 2–constrained).

The basic ingredients of the proof are:

(a) Fusion-theoretic; and

(b) 2–local theoretic.

By the latter term, we mean the analysis of the structure of the 2–local subgroups of G, particularly, the normalizers of elementary abelian 2–subgroups of G of maximal rank. However, now, because we are in a general classification problem, we have full information about the proper subgroup structure of G. In particular, their composition factors lie in the family $\mathscr{F}$ of simple groups listed in the conclusion of our theorem. We wish to emphasize that the analysis depends upon very detailed properties of the elements of $\mathscr{F}$. This is especially true of the components which occur in the centralizers of the involutions in cases (2) and (3).

For example, in the noncentral case (3), there is only a single simple group which satisfies the given conditions, namely, M_{12}. Therefore one must show somehow that the given assumptions force G to have a precise Sylow 2–group of order 64. This argument, which was carried out entirely by Harada, is unbelievably subtle and lengthy. The essential reason is that, although one knows the structure of the centralizer C of some noncentral involution very well at the outset, a Sylow 2–subgroup T of C may, a priori, have very small order compared to that of a Sylow 2–subgroup S of G. Yet it is the knowledge of T and C which must ultimately yield the structure of S.

5. Groups of component type. In an N-group, all 2–local subgroups are solvable and hence 2–constrained; so an N-group is necessarily of non-component type. Hence only the abelian and class 2 Sylow 2–group problems and the 3′–problem include groups of component type.

There have been essentially three distinct proofs of the abelian problem: Walter's original one, following the methods of the odd order and dihedral Sylow 2–group analyses; Bender's proof, which stems from his simplifications of the odd order uniqueness theorems and forms the basis of the "Bender" method; and finally a proof based on the signalizer functor theorem.

The Bender method has a decided advantage in this problem as it treats all 2–ranks (beyond 2) uniformly and does not distinguish between the even and odd component cases. Probably the same method can be adapted to the class 2 problem as well, but in general it seems to have an inherent limitation — or at least to lead one in a direction which appears to be more complicated than that of the signalizer functor method. This occurs because of the essential use that the Bender method makes of some form of the ZJ-theorem, which breaks down if the critical proper subgroups of G are not p-stable. In effect, this means that one would be left in general to analyze certain residual cases related to some fixed odd prime p (which ultimately would turn out to be the characteristic of the group one was classifying). On the other hand, the signalizer functor method avoids this problem by sticking closer to the centralizers of involutions and attempting to pin down their exact structures.

Since I wish later to relate the present discussion to future classification problems, I shall limit myself here to the signalizer functor approach. This requires us to divide the component type case into three parts: 2–ranks 3 and 4, odd component type, and even component type. The first of these is treated by the same general techniques which we described in the preceding section and in our discussion of Sylow 2–group characterization problems discussed in Part II; I shall say no more about these cases.

One further remark. In the 3′–problem, the family $\mathscr{F}$ consists only of the groups $Sz(2^n)$ and so a minimal counterexample is either of non-component type or of *even* component type. Hence only the abelian and class 2 Sylow 2–group problems include the case of odd component type.

a) *Odd component type.* We treat both problems simultaneously. I shall follow the argument that Gilman and I use to handle the class 2 case. We let G be a minimal counterexample and assume that G has 2–rank at least 5. Apart from the known groups with abelian Sylow 2–subgroups, the family $\mathscr{F}$ consists of the group $PSL(2,q)$, $q \equiv 7,9 \pmod{16}$, A_7, $Sz(2^n)$, $PSU(3,2^n)$, $PSL(3,2^n)$, and $PSp(4,2^n)$. Using properties of the elements of $\mathscr{F}$, we prove that G is 3–*balanced*. This means that for any elementary abelian 2–subgroup T of G of rank 3 and any involution x of G which centralizes T, we have

$$\Delta_G(T) \bigcap C_G(x) \subseteq O(C_G(x)),$$

where we have set

$$\Delta_G(T) = \bigcap_{t \in T^{\#}} O(C_G(t)).$$

Now let A be an elementary 2–subgroup of G of maximal rank. Then A has rank at least 5. Because of this, it follows from a general result, if for a in $A^{\#}$ we set

$$\theta(C_G(a)) = \langle C_G(a) \bigcap \Delta_G(T) \,|\, T \subseteq A, \text{rank of } T = 3 \rangle,$$

that

$$(*) \qquad \theta(C_G(a)) \bigcap C_G(b) \subseteq \theta(C_G(b))$$

for all a, b in $A^{\#}$.

Since each $\theta(C_G(a))$ is an A-invariant subgroup of $C_G(a)$ of odd order (actually $\theta(C_G(a))$ lies in $O(C_G(a))$ because of 3–balance), condition (*) asserts that θ is an *A-signalizer functor* on G.

Now the signalizer functor theorem yields that the group

$$W_A = \langle \theta(C_G(a)) \,|\, a \in A^{\#} \rangle = \langle \Delta_G(T) \,|\, T \subseteq A, \text{rank of } T = 3 \rangle$$

is of *odd* order.

The next step in the proof is to show, if $W_A \neq 1$, that $N_G(W_A)$ is strongly embedded in G, which, as usual, will yield a contradiction. It is in this argument that generational properties of the known simple groups, particularly groups of Lie type of odd characteristic, are needed. In the end, we conclude that $W_A = 1$ and consequently

$$(**) \qquad \Delta_G(T) = 1 \text{ for all subgroups } T \text{ of } A \text{ of rank } 3.$$

To describe the consequences of (**), we need some general terminology connected to the layer of a group X. Set $\bar{X} = X/O(X)$ and let $\bar{K}$ be a product of components of $L(\bar{X})$ with $\bar{K}$ normal in $\bar{X}$. Let K be the unique normal subgroup of X which is minimal subject to projecting on $\bar{K}$. Clearly K lies in the inverse image of $L(\bar{X})$. Now it may or may not happen that K itself is a semisimple group. It is easy to see that this will occur if and only if K centralizes $O(X)$. The special case that $\bar{K} = L(\bar{X})$ is of importance. If K is semisimple in this case, one says that X has a *semisimple layer*.

We are, however, here interested in the case in which K (or more precisely, $\bar{K}$ (modulo its center)) consists of those components of $L(\bar{X})$ which are elements of $\mathcal{F}$ of *odd* type. In this case, we set $K = \Lambda(X)$.

Then (**) has the following crucial consequence:

$$\Lambda(C_G(a)) \text{ is semisimple for every } a \text{ in } A^{\#}.$$

Using this conclusion, the next step in the proof is a demonstration that the group

$$V_A = \langle \Lambda(C_G(a)) \mid a \in A^{\#} \rangle$$

is a *nontrivial proper* subgroup of G.

Finally by an argument similar to that for W_A, we prove that $N_G(V_A)$ is strongly embedded in G.

Our argument thus yields that a minimal counterexample G is not of odd component type. This brief summary gives the main steps in the use of the signalizer functor method, as it applies in the odd component type case of the abelian and class 2 Sylow 2–group problems.

Finally we note that in the latter case A_7 is an element of $\mathcal{F}$ which is neither of odd or even type. However, in this particular problem one can include A_7 among the components of odd type without affecting the analysis. Hence in these two problems and in the 3′–problem, it remains to discuss the even component case.

b) *Even component type.* In this case it is easy to show that G is a balanced group. Hence by the Gorenstein-Walter-Aschbacher result, we have that $O(C_G(x)) = 1$ for every involution x of G. With A, as before, we now consider the subgroup

$$X_A = \langle L(C_G(a)) \mid a \in A^{\#} \rangle.$$

The goal of the analysis is to establish, as with V_A above, that X_A is a nontrivial proper subgroup of G. If this is the case, once again one derives a contradiction by showing that $N_G(X_A)$ is strongly embedded in G.

In proving that X_A is proper, a component L of maximal 2–rank of $L(C_G(a))$ as a ranges over $A^{\#}$ plays an important role. The argument is quite easy if the rank of A exceeds twice that of L. However, in the contrary case, it is very difficult, particularly in the 3′–problem. This is perhaps not surprising since the Rudvalis group contains a (noncentral) involution whose centralizer has the form $Z_2 \times Z_2 \times Sz(8)$. Of course, the Rudvalis group is not a 3′–group, but it is a general principle that whenever one is "close" to a real simple group—that is, whenever one is considering an extremal eonfiguration—thc analysis becomes very delicate. Dempwolf has recently characterized the Rudvalis group by the centralizer of such an involution.

In the abelian and class 2 Sylow 2–group problems, the main step in the proof consists in showing that

$$N_G(A) \subseteq N_G(L).$$

Once this is achieved, it is not difficult to show that X_A is semisimple and hence proper in G.

c) *Strongly closed abelian 2–subgroups.* Before going ahead, we must stop to describe an important extension of the abelian Sylow 2–group classification theorem due to Goldschmidt which we mentioned earlier. Ever since his dissertation, Goldschmidt has been interested in general questions about fusion of 2–elements in a simple group. Out of that initial analysis, he obtained the following striking result:

If a Sylow 2–subgroup S of a simple group has nilpotency class n, then the center of S has exponent at most 2^{n-1}.

Thus, in particular, if S has class 2, then $Z(S)$ must be elementary abelian.

Later he classified groups with a *weakly embedded* 2–local subgroup (a weakened form of the conditions for strong embedding, which I shall not define). It was this problem that led to his work on the signalizer functor theorem, for its solution required a signalizer functor result for abelian 2–subgroups of any rank at least 3.

In that problem, the abelian subgroup of concern was strongly closed in a Sylow 2–subgroup S of G. In general, a subgroup T of S is said to be *strongly closed* in S (with respect to G), if for any t in T, if t is conjugate in G to an element u of S, then necessarily u lies in T.

The culmination of Goldschmidt's work in this direction came in his recent classification of groups G in which a Sylow 2–subgroup contains a nontrivial strongly closed abelian subgroup A. The theorem gives a precise description of the normal closure of A in G. In particular, if G is simple, it either has abelian Sylow 2–subgroups (and so is known) or is isomorphic to $Sz(2^n)$ or $PSU(3,2^n)$. Glauberman's Z^*-theorem is just the special case that A has order 2. The theorem also generalizes a difficult result of Shult, which classifies groups in which the weak closure of an involution in its centralizer is abelian. The significance of Goldschmidt's result is that it provides a powerful new tool for studying the fusion of 2–elements in a simple group. It is used in the same way as Glauberman's Z^*-theorem, but now we can draw conclusions about an arbitrary abelian 2–subgroup rather than only a single involution.

The proof of the theorem can be viewed as a generalization of Bender's classification of simple groups with abelian Sylow 2–subgroups. Although Goldschmidt is forced to treat more complex configurations, the essence of his argument consists in establishing the same kind of uniqueness results that Bender uses and accomplishing this by the Bender method. As Goldschmidt points out, if A has rank at least 4, then he could have used the signalizer functor method instead. This would have left him with the low rank cases of A to treat by a separate argument. However, the Bender approach allows him to give a uniform argument that covers all cases simultaneously.

6. Groups of non-component type of low 2-local p-rank, p odd. In the abelian and class 2 Sylow 2–group problems, groups of non-component type are treated uniformly and it is not necessary to separate out the case of low 2–local p-rank. Thus we are concerned here with N-groups and $3'$–groups together with Janko's analysis of thin groups with solvable 2–local subgroups.

In the case of N-groups and $3'$–groups, subcases must be considered according as G is a thin group or has 2–local p-rank equal to 2 for some odd prime p. The latter case requires a combination of the methods used in the thin case and in the large 2–local p-rank cases. One wishes to construct uniqueness subgroups for odd primes, but because a Sylow p-subgroup of G may have p-rank 2, rather than at least 3, one must play off the p-structure against the 2–structure to carry this out[†]. This is all I shall say about this situation and shall limit myself now to a brief discussion of the thin case.

There are three major components to the analysis:

a) Factorization lemmas for the prime 2;

b) Weak closure arguments involving abelian 2–subgroups; and

c) Analysis of the centralizers of elements of odd prime order.

Here is an example of a factorization lemma. Let X be a solvable group of order prime to 3 with $O(X) = 1$ and let T be a Sylow 2–subgroup of X. Set $Z(T)$ = center of T and $J^*(T)$ = subgroup of T generated by the abelian (or elementary abelian) subgroups of T of maximal rank. (The subgroup J of the

[†] F. Smith has recently extended Thompson's N-group analysis of this case, assuming only that G has solvable 2-local subgroups.

Glauberman ZJ-theorem is defined instead by the abelian subgroups of T of maximal *order*.) Then we have

$$(*) \qquad X = X_1X_2,$$

where

$$X_1 = C_X(Z(T)) \text{ and } X_2 = N_X(J^*(T)).$$

If T had been a Sylow p-subgroup of X for an odd prime and X had been p-stable, the ZJ-theorem would have asserted that $Z(J(T))$ is normal in X, whence $X = N_X(Z(J(T))$. Such a result would be far preferable to the factorization (*), but unfortunately one has been unable to obtain any direct analogue of the ZJ-theorem for the prime 2. Factorization lemmas are the only substitute which we have, but they are much more difficult to apply effectively.

The way in which a factorization lemma is used is roughly as follows. If X is a 2–local subgroup of the group G and we set $N_1 = N_G(Z(T))$ and $N_2 = N_G(J^*(T))$, then $X_1 \subseteq N_1$ and $X_2 \subseteq N_2$. Thus a knowledge of the structure of the two specific 2–local subgroups N_1 and N_2 yields information about the more arbitrary 2–local subgroup X. Moreover, if one can show that $N = N_1N_2$ is a proper subgroup of G, we see that X is contained in N, which is the kind of result needed to prove that N is strongly embedded in G.

Again let X and T be as above, but assume only that X is 2–constrained (with $O(X) = 1$). Thompson has shown that there is a unique abelian normal 2–subgroup V of X with the property that $\bar{X} = X/C_X(V)$ has no nontrivial normal 2–subgroups. Because X is 2–constrained, V contains $Z(T)$; so in the applications $C_X(V)$ lies in the subgroup N_1. Hence, in some sense the structure of X is determined from that of N_1 and the action of the group $\bar{X}$ on V. Here $\bar{X}$ is isomorphic to a subgroup of $\mathrm{Aut}(V)$, so V acts as a module for X.

One is interested in the *weak closure* V^* of V in T with respect to G, that is, the subgroup of T generated by all conjugates of V in G which lie in T. If $V^* \subseteq C_X(V)$, then we have a factorization $X = C_X(V)N_X(V^*)$ by the Frattini argument, so $X \subseteq N_1N_G(V^*)$. This again allows one to pursue certain lines of argument. Hence the case of primary concern occurs when $V^* \nsubseteq C_X(V)$. This means that $\bar{V}^*$ act faithfully on V and Thompson studies this action. Eventually he shows that the structure of V and, in particular, its order are very restricted.

Let me just say in conclusion that if M is a maximal 2–local subgroup of G containing T, one must also consider the action of subgroups P of prime order p, p

odd, on $O_2(M)$. A delicate interplay of the methods we have touched on allows Thompson to infer that P has few (and often no) nontrivial fixed points on various subgroups and sections of $O_2(M)$ on which P acts.

I want to emphasize the unbelievable subtlety and complexity of these arguments. At the outset, a Sylow 2–subgroup of G has completely arbitrary order. Yet in the end Thompson has to show that G is isomorphic to the Tits simple group. (The Ree group of characteristic 2^n is simple only if $n > 1$. When $n = 1$, Tits showed that it has a subgroup of index 2 which is simple. Hearne showed that the Tits group is, in fact, an N-group. Parrott has characterized the Tits groups by the structure of the centralizer of an involution.) Thus the analysis must somehow yield that a Sylow 2–subgroup of G has order 2^{11} and that G has precisely two conjugacy classes of maximal 2–local subgroups: one of order $2^{11}\cdot 3$ and the other of order $2^{11}\cdot 5$.

7. Groups of non-component type. Finally we consider the case in which G is a general group of non-component type. Clearly we can assume that G does not possess a strongly embedded subgroup or else Bender's theorem yields the possibilities for G. This immediately disposes of the abelian Sylow 2–group problem, for under the present assumptions the normalizer of a Sylow 2–subgroup S of G is strongly embedded.

In the class 2 Sylow 2–group problem, the aim is to prove that $G \cong PSL(3,2^n)$ or $PSp(4,2^n)$ for some $n \geqq 2$. Unfortunately the methods which achieve this are very special and give no insight into how to attack more general problems. In view of Goldschmidt's theorem, one can assume also that no nontrivial abelian subgroup of S is strongly closed in S. Under this assumption, one analyzes the subgroup $J^*(S)$ generated by the elementary abelian subgroups of S of maximal rank and its embedding in 2–local subgroups of G containing S. (It is easy to see in the present case that every 2–local subgroup contains a Sylow 2–subgroup of G.) Ultimately one is able to pin down the exact structure of all maximal 2–local subgroups of G and, in particular, of S. It turns out that there are exactly two maximal 2–local subgroups M_1 and M_2 of G containing S and that S is isomorphic to a Sylow 2–subgroup of $PSL(3,2^n)$ or $PSp(4,2^n)$.

In both cases, one can then prove that $G_0 = \langle M_1, M_2 \rangle$ is a (B,N)-pair, whence $G_0 \cong PSL(3,2^n)$ or $PSp(4,2^n)$, as the case may be. Now the strong embedding theorem forces $G = G_0$. However, as Collins has already characterized $PSL(3,2^n)$ by its Sylow 2–subgroup, it is quicker to quote his result in the first case.

Thus our main interest is in the argument that Thompson uses to deal with the case that G is either an N-group or a $3'$–group. In each case, the goal of his analysis is to derive a contradiction, for there exist no simple groups satisfying these conditions. (We are assuming, of course, that the 2–local p-rank of G is at least 3 for some odd p.) The essential tool in reaching this conclusion is the establishment of a uniqueness theorem for such odd primes p. We have alluded to this notion several times before without specifically defining the term, but now it is time to say something about it.

A *uniqueness subgroup* for p, p odd, is simply the analogue for p of a strongly embedded subgroup for the prime 2. Indeed, if P is a Sylow p-subgroup of G with $P \neq 1$, let us say that a proper subgroup M of G is *strongly p-embedded* in G provided the following conditions hold:

a) $C_G(x) \subseteq M$ for any element x of M of order p;

b) $N_G(P) \subseteq M$.

This is the precise analogue of a strongly embedded subgroup. The point is that a uniqueness subgroup is defined by conditions slightly weaker than (a) and (b). In practice, it turns out to be easier to construct uniqueness subgroups for p than strongly p-embedded subgroups. On the other hand, if P has sufficiently high rank, a uniqueness subgroup for p will necessarily be strongly p-embedded. Hence, for simplicity, let us act in this discussion as though the two concepts were identical.

It would be very nice if we had the analogue of Bender's theorem and could assert that a simple group G of p-rank at least 3 with a strongly p-embedded subgroup M had to be isomorphic to $PSL(2, p^n)$ or $PSU(2, p^n)$, or in the case $p = 3$, to a group of Ree type of characteristic 3. Unfortunately no such theorem exists nor does it seem likely that we shall soon attain it. Thus some other device must be used to reach a contradiction.

Keep in mind that the prime p is not arbitrary, but has been chosen in relation to the 2–local structure of G. One can therefore reasonably ask: can we prove that our strongly p-embedded subgroup M is strongly embedded in the ordinary sense?

Let me give a quite general set of conditions under which this is indeed true. First of all, it is not difficult to prove the following general result which I have referred to as the *theorem of transition.*

THEOREM. *If each maximal 2–local subgroup of G lies in some conjugate of a fixed proper subgroup K of G, then K is strongly embedded in G.*

Thus, by this result, we need only give conditions which imply that every maximal 2–local subgroup of G lies in a conjugate of our strongly p-embedded subgroup M. Such a set of conditions is the following.

For any maximal 2–local subgroup H of G, we have

a) The p-rank of H is at least 2;

b) H does not contain a strongly p-embedded subgroup.

Of course, these two conditions need not hold for *every* maximal 2–local subgroup of G. On the other hand, in the N-group and $3'$–problems, they do hold at least for a uniqueness subgroup M for the prime p, which turns out in both problems to be a maximal 2–local subgroup.

To deal with the various possibilities that can arise, Thompson must again investigate the weak closure of certain abelian 2–subgroups of G. In the $3'$–problem, he is eventually able to show in all cases that M is strongly embedded in G. However, in the N-group problem, two residual cases remain which require a separate analysis.

a) $O_2(M)$ is of symplectic type (that is, it has no noncyclic characteristic abelian subgroups, equivalently, is the central product of an extra-special group with either a cyclic, dihedral, quasi-dihedral, or generalized quaternion group; an extra-special 2–group being one of class 2 whose center, derived group, and Frattini subgroup are all equal and of order 2).

b) M contains no normal elementary abelian 2–subgroups of order exceeding 4, M contains an elementary abelian 2–subgroup V of order 4, and $C_G(v) \subseteq M$ for each v in $V^{\#}$.

Thompson shows that there exists no simple N-group with $SCN_3(2)$ nonempty which possesses a maximal 2–local subgroup M satisfying either the conditions of (a) or (b). (This argument does not require M to be a uniqueness subgroup for some odd prime p.) We should point out that this result is used by Thompson, not only to deal with the above residual cases concerning N-groups of high 2–local p-rank, but are also invoked in treating N-groups of low 2–local p-rank for all odd p. We also remark that Lundgren and Pomareda, in their doctoral theses, under the supervision of Janko, have extended these two results of Thompson for the N-group case to the case in which G is assumed only to have all of its 2-local subgroups solvable (rather than all of its local subgroups solvable).

Finally we note that in the N-group case, Thompson's construction of uniqueness subgroups for odd p is modeled after the corresponding constructions in the odd order paper; while in the case of 3′–groups, it depends very heavily upon the fact that 3′–groups of order prime to 5 are solvable groups to which the so-called "three-against-two" factorization arguments can be applied. Hence it would seem that neither of these constructions is amenable to very extensive generalization. As we shall discuss in Part IV, the signalizer functor method (for odd primes) would seem to provide a uniform procedure for carrying out such constructions in more general classification problems.

IV. The Prospects for Classifying Simple Groups

The development of the theory of finite groups, in particular the simple groups, was very slow during the first half of the twentieth century: Frobenius, Burnside, Dickson, Blichfeldt, Schur, Philip Hall, Brauer, Wielandt. The first true group-theoretic classification theorem concerning simple groups was due to Zassenhaus in the middle 1930's. These are almost the only names that come to mind. In the 1950's, from the influence of Hall and Brauer, there came a renewed interest in the entire subject. But then, under the impact of the Feit-Thompson odd order theorem, this interest was greatly intensified and broadened, bringing in its wake the tremendous surge of research activity of the 1960's and 70's.

I have tried to describe the scope of the achievement of the past 15 or 20 years—how far we have moved in the study of simple groups in that short period from the first classification theorems and the proof of the solvability of groups of odd order. It remains only to relate this accomplishment to the ultimate problem of classifying all finite simple groups. Have all our efforts been just the first feeble probings? Have we uncovered only small sporadic groups because we are still at the edge of the problem? Must we yet discover whole new infinite families of simple groups? Or have we passed through the roughest part of our journey and with the methods we have developed reach our final objective? What then are the prospects for the future?

My own view is closer to the second position than to the first, for it is my strong conviction that the basic four-part subdivision which emerged from our analysis of the general classification theorems already established provides a firm foundation for the further study of simple groups; and, moreover, effective general techniques now exist for analyzing the problems that will arise in each part.

This does not mean that I feel that our present list of simple groups is complete and that no new sporadic or even infinite families of groups remain to be discovered. The given subdivision does not justify such a belief. What it means rather is that one can expect our approach to simple groups in the years ahead to be much more systematic than in the past.

The way to view the subdivision is as follows. The classification of simple groups has a fundamental breakup into two major cases:

I) Groups of component type;

II) Groups of non-component type.

This is equivalent to the following subdivision:

I) Groups in which some 2–local subgroup is not 2–constrained;

II) Groups in which all 2–local subgroups are 2–constrained.

Furthermore, within each category, one must treat the cases of "small" groups separately by special methods.

Thus if we set aside the low rank cases, we anticipate then that the analysis of simple groups of component type within a general classification problem will lead to characterizations of families of groups of Lie type of odd characteristic, alternating groups, and some sporadic groups; while the corresponding analysis for groups of non-component type will lead to characterizations of families of groups of Lie type of even characteristic and the remaining sporadic groups.

In the four general problems so far treated — groups with abelian and class 2 Sylow 2–subgroups, N-groups, and $3'$–groups — what families of groups were we led to ? In the component type case, the answer is none; in these cases the families $\mathscr{F}$ were so restricted that every configuration led to a contradiction (assuming 2–rank at least 4). On the other hand, in the non-component type cases, we were led to the families $PSL(2,2^n)$, $PSU(3,2^n)$, $Sz(2^n)$, $PSL(3,2^n)$, or $PSp(4,2^n)$. However, the first three families arise only when G has a strongly embedded subgroup and the last two only in the class 2 Sylow 2–subgroup problem. But certainly we shall assume in our analysis that our group G does not possess a strongly embedded subgroup. Moreover, we have remarked earlier that the argument that leads to the groups $PSL(3,2^n)$ and $PSp(4,2^n)$ in the class 2 problem is very special and not capable of much generalization. The inescapable conclusion is this: the four problems so far treated have not been broad enough to yield a general method for obtaining characterizations of families of simple groups.

However, as I shall indicate, this is not the fault of the methods we have developed, but of the problems we have so far attacked! Presently I shall explain how our methods wil work when we come to deal with broader classification problems.

1. Groups of low 2-rank. Beyond the results already proved, we probably shall require only an analysis of groups of component type of 2–rank 3 and 4. The reason for this is that, on the one hand, the general case of groups of component type seems to begin in 2–rank 5; and on the other, groups of non-component type of 2–rank 3 or 4 have low 2–local p-rank for all odd p and so it will probably be better to treat this problem as part of the broader, latter problem.

The methods developed in the classification of groups of sectional 2–rank at most 4 will undoubtedly provide the basis for the analysis. Thus there is not much more to say about this problem; but just to give a little bit of the flavor, let me mention two extremal situations that the sectional 2–rank analysis clearly indicates will have to be considered. For some noncentral involution x of G,

$$C_G(x) \cong Z_2 \times \mathrm{Aut}(PSL(2,9)) \text{ or } Z_2 \times \mathrm{Aut}(PSL(3,3)).$$

The first case actually occurs in the Higman-Sims group. This group has 2–rank 4, but *sectional* 2–rank 5, so it did not enter into the sectional 2–rank 4 analysis. At present, it has been characterized by the centralizer of a *central* involution, but not by the centralizer of a noncentral involution. What is interesting is that the 2–structure and fusion of involutions is very similar in the two centralizers listed. Presumably the second one cannot arise in a simple group, but that remains to be demonstrated[†].

2. Groups of low 2-local p-rank, p odd. In contrast to the low 2–rank situation, this whole problem area is only now beginning to be investigated and it is not yet even clear precisely what it should include. My own guess is that the proper analogue of the low 2–rank problem is the following:

Determine all simple groups of non-component type whose 2–local 3–*rank* is at most 4.

I shall comment on this formulation in Section 3.

Whatever the ultimately correct statement of this problem area turns out to be, one thing is certain, a great deal of work will be involved in its complete resolution. Certainly all the methods that Thompson developed for the corresponding cases of the N-group and $3'$–problems will form the basis for the analysis. Very likely the most difficult subcase of all will be that of thin groups.

[†] This result has now been established by Griess.

One reason for the difficulty will be the fact that quite general configurations of 2–local subgroups may exist in which all factorization lemmas fail. In the N-group problem, this could also happen in a 2–local subgroup H of order divisible by 3 (if H had a section isomorphic to $PSL(2,2) \cong S_3$); but there it was possible to "localize" the problem. In the more general situation in which H is non-solvable (and 2–constrained), the presence of sections isomorphic to $PSL(2,2^n)$, n arbitrary, can cause factorization failure, but now it may be more difficult to pin things down.

3. The primes 2 and 3. It remains only to comment on groups of component and non-component type (of sufficiently high rank). It is my feeling that a considerable parallel exists between these two cases, and that the same *signalizer functor* and *Bender* methods can be used to treat both problems, the distinction being the following:

In groups of component type, study centralizers of involutions. In groups of non-component type, study centralizers of elements of order 3.

This may seem like a very surprising assertion to make, but a little reflection about the groups of Lie type will show that it is rather natural. Indeed, in the groups of Lie type of odd characteristic, an involution is a "semisimple" element (in the Lie sense) and its centralizer is dominated by its layer. However, the same conclusions are true for elements of order 3 in groups of Lie type of even characteristic. Of course, they are also true for elements of *arbitrary* odd prime order p, so why single out the prime 3? The answer is that the layers of the centralizers of elements of order 3 will be larger than those of elements of order $p > 3$; and so they constitute a larger "chunk" of the given group. Moreover, involutions lie in a Cartan subgroup, in general, in odd characteristic; while elements of order 3 are very "close" to lying in a Cartan subgroup in even characteristic, at worst we may require a quadratic extension of the underlying field to achieve this; whereas for elements of order $p > 3$, larger field extensions may be needed.

In the case of groups of component type, the procedure we outlined in the preceding part was to study the centralizers of involutions by means of the signalizer functor method, the goal being to rule out configurations by constructing strongly embedded subgroups.

The analogy then is that for groups of non-component type, one should study he centralizers of elements of order 3 by the same method, obtaining corresponding contradictions by constructing *strongly* 3-*embedded* subgroups. As we indicated in the preceding part, the existence of a strongly 3-embedded subgroup M does not automatically yield a contradiction—in fact, the process of proving that M is actually strongly embedded in G may require some fairly elaborate weak closure arguments involving abelian 2-groups as well as an independent analysis to handle certain residual cases in which $O_2(M)$ is of a restricted shape and the centralizers of certain involutions of $O_2(M)$ lie in M. However, we should not lose sight of the importance of constructing strongly 3-embedded subgroups just because difficult obstacles may yet block the path to deriving final contradictions.

We wish therefore to ask to what extent it might be possible to carry over the signalizer functor method for involutions to elements of order 3. Now in the case of involutions, successful application of the method depends primarily upon two things:

a) The signalizer functor theorem; and

b) Balance and generational properties of the elements of $\mathscr{F}$ related to the prime 2.

Hence what we are really asking is to what extent can we carry over (a) and (b) to the case of elements of order 3 and elementary abelian 3-groups? First of all, Goldschmidt's general signalizer functor theorem applies to any *solvable* A-signalizer functor θ, where A is an abelian p-subgroup of G of rank at least 4 and p is an arbitrary prime†. Solvable here means that each of the A-invariant p'-subgroups $\theta(C_G(a))$ of $C_G(a)$ is solvable for a in $A^{\#}$. (In the case $p = 2$, this was automatically true as every $2'$-group, being of odd order, is solvable.) Thus a new problem arises since we may be forced to consider nonsolvable signalizer functors. But here is another advantage in working with the prime 3 rather than other odd primes p: the only nonsolvable composition factors of the $3'$-groups $\theta(C_G(a))$ will, by Thompson's theorem, be Suzuki groups. For no other value of p do we have a comparable result. We note also that the fact that G is of non-component type will very likely greatly restrict the possible nonsolvable signalizer functors that can actually arise.

As far as balance and generational properties of the elements of $\mathscr{F}$ relative to the prime 3 are concerned, one can anticipate that the situation will be entirely analogous to that for the prime 2.

† Glauberman has now extended this result to the rank 3 case.

There is one other problem that must be mentioned. When the centralizer of every involution of G is 2-constrained, the signalizer functor method only goes so far as to prove that $O(C_G(x)) = 1$ for every involution x of G. At that point, we are forced to turn our attention to odd primes. Hence a similar situation will arise if the centralizer of every element of order 3 is 3-constrained, namely, the most that the signalizer functor method can yield for the prime 3 is that $O_{3'}(C_G(x)) = 1$ for every element x of order 3 in G. In other words, in this case the signalizer functor method is not strong enough to yield a strongly 3-embedded subgroup.

The fact is that this special case constituted an important subproblem within the N-group analysis. Thompson treated it independently and was led to very beautiful characterizations of the two simple groups $G_2(3)$ and $PSp(4, 3)$.

Fortunately this problem is not as serious as first appears. Indeed, it seems that this case can arise only under the following conditions†:

If M is a maximal 2-local subgroup of G of maximal 3-rank (at least 3), then $O_2(M)$ is necessarily of symplectic type.

But this is just one of the residual cases that we would need to treat, in any event, even after we had constructed a strongly 3-embedded subgroup. This is another indication that these residual cases that occur when $O_2(M)$ has a restricted structure represent an intrinsic subcase of the general problem of classifying groups of non-component type.

Our discussion thus shows that there is strong reason to believe that the whole signalizer functor machinery will eventually carry over for the prime 3 to the general study of simple groups of non-component type.

4. Groups of component type. So far one has only had to consider groups of *odd* and *even* component type, but it is very likely that in more general problems, one will be forced to treat three possibilities separately: odd, *alternating*, and even type.

We shall focus here on the odd component type case. If we go back to the abelian and class 2 Sylow 2-group discussion, the main steps of the argument go as follows:

a. 3-balance;
b. Construction of $\theta(C_G(a))$ for a in $A^{\#}$;
c. W_A of odd order;

† Geoffrey Mason has recently completely solved this problem.

d. $N_G(W_A)$ strongly embedded;
e. $W_A = 1$, so all $\Delta_G(T) = 1$;
f. $\Lambda(C_G(a))$ semisimple for a in $A^{\#}$;
g. V_A nontrivial and proper;
h. $N_G(V_A)$ strongly embedded.

Where then is the weak link in this chain? First of all, if $\mathscr{F}$ is the family of all known simple groups, it is almost always true that our group G will be 3–*balanced*; this will only fail in the presence of components in the centralizer of some involution isomorphic to A_n or $\hat{A}_n$ with $n \equiv 3 (\text{mod } 8)$. In the first case, one can construct the identical A-signalizer functor as before. However, even in these exceptional cases. one can prove that G is 4–balanced relative to A, which allows us to construct an A-signalizer functor θ on G using T's of rank 4 rather than rank 3.

This would indicate that we ought to be able to show in general that the groups $\Lambda(C_G(a))$ are semisimple for all a in $A^{\#}$. Indeed, this is a natural objective since it holds in the groups of Lie type of odd characteristic. Moreover, John Walter and I have established just such a result under a very general set of assumptions on a simple group G (of suitably high 2–rank). Thus the chain appears to be strong through step (f). Furthermore, *if* we can prove that the subgroup V_A is a nontrivial proper subgroup of G, then the same argument that yielded $N_G(W_A)$ to be strongly embedded in G will prove that $N_G(V_A)$ is as well. The nontriviality of V_A is obvious if some $\Lambda(C_G(a)) \neq 1$, which in general will be the case. Thus the break must occur in trying to prove that V_A is a *proper* subgroup of G. Indeed, if G is of Lie type of odd characteristic and A is an abelian 2–subgroup of G of maximal rank, one has $V_A = G$ with the exception of certain groups of low Lie rank.

We see then the single new problem which one will face in general which did not arise at all in the abelian and class 2 Sylow 2–group problems:

What are the consequences of the assumption $V_A = G$?

In other words, one will try to prove that V_A is a proper subgroup of G and pin down the precise conditions under which the argument breaks down. Our work with Walter (and similar recent results of Aschbacher) indicates that the conditions are the following:

The centralizers of the involutions of A resemble those in a group of Lie type of odd characteristic.

In particular, for some a in $A^{\#}$, $H = C_G(a)$ contains a quasisimple normal subgroup L of Lie type of odd characteristic such that

$$C_H(L) \text{ has 2-rank } 1.$$

This means that $C_H(L)$ is very small. Since $H/C_H(L)$ is isomorphic to a subgroup of Aut(L), we see that L completely dominates the structure of H. For brevity, we say that H is in *standard form.*

The fact is that every group of Lie type of odd characteristic, again with the usual exceptions, possesses an involution whose centralizer is in standard form.

This then is the upshot of the discussion: Successful application of the signalizer functor method to groups of odd component type in a given general classification problem will lead to an *extended centralizer of an involution* problem, that is, one will be reduced to characterizing the groups of Lie type of odd characteristic by the centralizer of an involution in standard form.

We should say that the extended centralizer problems so far considered have, in effect, treated only special cases of the standard form problem. Hence further extensions of those results will be required for the ultimate determination of the simple groups of odd component type.

A similar discussion (which we shall present only very briefly) applies to groups of alternating and even component type. In these cases, one can not expect quite such sharp results, but only the following.

For some involution a in $A^{\#}$, $H = C_G(a)$ contains a quasisimple normal subgroup L isomorphic to a group of Lie type of even characteristic or to a central extension (possibly trivial) of an alternating or sporadic group such that

$$C_H(L) \text{ has 2-rank at most that of } L.$$

For brevity, we say that H is in *quasistandard* form under these conditions.

Aschbacher has attempted to put a large portion of this analysis in a general framework. Essentially he begins at the point in which the centralizer of every involution has a semisimple layer and goes on from there to pin down the structure of the centralizers of involutions[†]. Of course, if the group G is of *even* component type, then in almost all cases, G would be a balanced group, in which case $O(C_G(x)) = 1$ for every involution x of G. Obviously under these conditions the centralizer of every involution does have a semisimple layer.

We remark that in the known simple groups of component type, it appears to

[†] It is now evident that Aschbacher's result that $C_H(L)$ is *tightly embedded* in G is fundamental. His recent work strongly indicates that groups of component type in which centralizers of involutions have semisimple layers may be classified completely in the not too distant future.

be the case that the involution a can always be chosen so that $C_H(L)$ has 2–rank at most 2. However, except in the case of *odd* component type, the initial analysis would not seem to yield such a sharp conclusion.

In any event, the discussion indicates that we shall need characterizations of alternating and sporadic groups of component type by the centralizers of involutions in quasistandard form. In the case in which L is of Lie type of even characteristic (apart from the groups $PSL(2,4) \cong PSL(2,5) \cong A_5$, $PSL(3,2) \cong PSL(2,7)$, and $PSL(4,2) \cong A_8$ and their covering groups) any simple group G having the centralizer of an involution of such a form would necessarily be a *sporadic* group. Hence the study of simple groups in which the centralizer of some involution is in quasistandard form with component of Lie type of even characteristic will provide both a systematic way of searching for new sporadic groups as well as an essential and necessary step for the ultimate classification of simple groups of component type.

Let me conclude by mentioning a recent result of Harris and myself which illustrates the broad power and effectiveness of the signalizer functor method. I have earlier noted the way in which characterizations of direct products of simple groups enter into Sylow 2–group classification theorems of simple groups. Harris and I have attempted to put this whole problem area into a general setting.

Thus we consider an arbitrary group G with Sylow 2–subgroup S of the form $S = S_1 \times S_2$ such that both S_1 and S_2 are strongly closed in S (that is, no element of S_i is conjugate in G to an element of $S - S_i$, $i = 1,2$). Under the single assumption that the 2-local subgroups of G statisfy a certain condition which is expressed entirely in terms of the notion of 3–balance (and which allows for the exceptional nature of the groups A_n, $n \equiv 3 \pmod 8$)), we prove that G necessarily contains normal subgroups G_1, G_2 with Sylow 2–subgroups S_1, S_2 respectively. In particular, if $O(G) = 1$ and G has no proper normal subgroups of odd index, then $G = G_1 \times G_2$. The signalizer functor theorem and the signalizer functor method provide the essential tool in the proof of this general result.

5. Groups of non-component type. If the signalizer functor method can be used for the prime 3 in the study of simple groups of component type of sufficiently high 2–local 3–rank as effectively as we hope that it will work for groups of non-component type, one will be led in the same way to *extended centralizer of elements of order* 3 problems of the same general shape as the extended centraliser of involution problems we have just described. However, because we are restricted to groups of non-component type, it is very likely that we shall be limited primarily to the case in which the corresponding quasisimple normal subgroup L of the centralizer H of the element a of order 3 is of Lie type of even characteristic or a sporadic group and $C_H(L)$ has 3–rank 1, thus to the case that H is in *standard form* for the prime 3. Thus at the least we shall require characterizations of the groups of Lie type of even characteristic and of sporadic groups of component type by centralizers of elements of order 3 in standard form.

As we have mentioned in our discussion of special classification problems, only scattered results have so far been obtained in this whole area.

6. In conclusion. Is it possible that the preceding discussion provides a long-range program for determining all finite simple groups? The answer to this question depends really upon two things:

a) The number and complexity of simple groups not yet discovered;

b) The existence of subcases of general classification problems in which our present methods are ineffective.

My own deep belief is that our present methods, even if not capable of carrying us to our ultimate objective, are nevertheless strong enough to handle very large areas of the total problem, including what one may call the general cases and eventually will sharply delineate certain precise residual areas — some intractable "pockets of resistence" — on which future efforts will have to be focused.

My feeling is that these difficult areas are more related to groups of noncomponent type than they are to groups of component type and that, in fact, our present methods appear to be strong enough to deal with all aspects of the problem of classifying simple groups of component type[†]. If one looks back on how much has been accomplished in the past 15 years, it seems well within the realm of possibility that the next 15 years will see the complete determination of all simple groups of component type. This would represent a great achievement, for it would characterize for all time the groups Lie type of odd characteristic, the alternating groups, and certain of the sporadic groups (including possibly some as yet undiscovered ones). As our basic subdivision suggests, such a result would constitute

[†] The central question still outstanding is whether in an *arbitrary* simple group G the centralizer of every involution must have a semisimple layer.Thompson has recently made significant progress on this basic conjecture and there are indications that a combination of the Bender and signalizer functor methods together with the determination of all quadratic pairs and some special classification theorems may ultimately yield an affirmative answer.

the halfway mark in the complete classification of all finite simple groups. Perhaps, too, the next 15 years will see our understanding of groups of noncomponent type reach our present level of knowledge of the groups of component type, thus putting us within striking distance of our ultimate objective.

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